Proposition de parcours MA2 2027-2028

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1 Probability and statistics

This program consists of two intertwined series of courses in probability (P) and statistics (S), the aim of which is to provide solid and interdisciplinary teaching in modern theoretical aspects of the mathematics of randomness. Probability and statistics are represented in a balanced way, and a number of topics at the interface between several fields are offered (EDP, mathematical physics, neural networks).

Students will choose 3 courses out of the 4 offered in the first semester (P1, P2, S1, S2) and 4 courses out of the 8 offered in the second semester (P3-P6, S3-S6). It is also possible to take courses in other courses, particularly in PDE.

Note that four of the statistics-oriented courses (one in the first semester and three in the second) are offered by the M2 *Maths in action* program.

Finally, it is strongly recommended to take refresher courses.

2 Refresher Course

Stochastic tools (Bruno Schapira, 15h)

1. Discrete time martingales: stopping theorems and convergence. Extensions for continuous time martingales.

2. Construction of Brownian motion. Regularity of trajectories.

3. Some properties of Brownian trajectories. Connection with the heat equation.

3 Courses

3.1 First semester

P1 : Stochastic calculus (Cédric Gerbelot, 24h)

This lecture series will present some of the most important tools allowing to build and study continuous-time stochastic processes, the central example of which is of course Brownian motion. To this end, we will have to introduce and study semimartingales, a rich class of processes for which one can develop a differential and integral calculus, and set and solve certain type of stochastic differential equations.

Just as for the familiar ordinary differential equations (or PDEs), the motivation to study such stochastic differential equations comes from the goal of understanding the global behaviour of random processes by equations describing their infinitesimal behaviour. But since we are dealing with random processes, these equations contain a random "noise", which informally is an infinitesimal increment of Brownian motion. The main problem of their study comes from the fact that Brownian motion (and therefore the other processes of interest) have too rough trajectories (nowhere differentiable, for instance) for the usual differential and integral calculus to make sense.

In front of this obstacle, we will develop a notion of stochastic integral, due to Itô. It will give rise to a particular integral calculus, in which Itô's formula acts as an integration by parts (or a fundamental theorem of analysis) of a new kind. This integral calculus will allow us to study the stochastic differential equations for continuous semimartingales, and will shed a new light on these processes, for instance via Lévy's characterization of Brownian motion, or the Dubins-Schwarz theorem according to which continuous martingales are appropriate time-changes of Brownian motion. Contents:

1. Generalities on continuous-time processes
2. Continuous-time martingales, regularization. Local martingales, semimartingales. Bracket of a continuous semimartingale.
3. Stochastic integral with respect to a continuous semimartingale.
4. Itô's formula and applications. The theorems of Lévy, Dubins-Schwarz, Girsanov. Burkholder-Davis-Gundy inequalities.
5. Stochastic differential equations. The Lipschitz case.
6. (Time allowing) Continuous-time Markov processes. Generators. Diffusions.

References:

[1] Karatzas-Shreve: Brownian motion and stochastic calculus
[2] Le Gall: Brownian motion and stochastic calculus
[3] Mörters-Peres: Brownian motion
[4] Revuz-Yor: Continuous martingales and Brownian motion
[5] Varadhan: Stochastic processes

P2 : Statistical physics (Christophe Garban, 24h)

In statistical physics, one is interested in physical models made of a large number of microscopic elements which interact together in a simple fashion. The goal is then to understand how come such simple microscopic mechanisms can generate interesting (and surprising!) macroscopic phenomena such as phase transitions or symmetry breaking. This program has lead to the development of an important branch of probability theory and the aim of this course is to give a panorama of the field together with tools and techniques that are used in statistical mechanics. We shall focus on three fundamental models: percolation, Ising model and O(n) spherical spin model. Program of the course:

1. Percolation
    - Definition, phase transition
    - FKG inequality, $p_c = 1/2$
    - Exponential decay in the sub-critical regime
    - Russo-Seynour-Welsh theorem for critical percolation
2. Ising model
    - Definition, correlation inequalities
    - Infinite volume limit, Free energy, phase transition
    - Low temperature and Peierls argument
    - Uniqueness at high temperature
3. Spin models with continuous symmetry (for example, spins with values in spheres S^d)
    - No symmetry breaking in dimension 2 (Mermin-Wagner theorem)
    - Gaussian Free Field
    - Symmetry breaking in $\mathbb{Z}^d$, $d \geq 3$

References :

[1] W. Werner, Percolation et modèle d'Ising, Soc. Math. France, 2009.
[2] Y. Velenik, Introduction aux champs aléatoires markoviens et gibbsiens, http://www.unige.ch/math/folks/velenik/Cours/2006-2007/Gibbs/gibbs.pdf.

S1 : Concentration of measure in probability and high-dimensional statistical learning (Aurélien Garivier 24h)

This course will introduce the notion of concentration of measure and highlight its applications, notably in high dimensional data processing and machine learning. The course will start from deviations inequalities for averages of independent variables, and illustrate

their interest for the analysis of random graphs and random projections for dimension reduction. It will then be shown how other high-dimensional random functions concentrate, and what guarantees this concentration yields for randomized algorithms and machine learning procedures to learn from large training collections. This course will be based on a sample of the classical textbook "Concentration Inequalities" by Boucheron, Massart, Lugosi, and on "High-Dimensional Probability" by Roman Vershynin. Applications to machine learning will rely on "Understanding Machine Learning", by Shalev-Schwartz and Ben-David.

S2 : Stochastic modeling and statistical learning (Romain Azaïs, Nicolas Jouvin, Clément Marteau, 24h)

[Also in the M2 *Math en Action* program]

- High-dimensional regression: Concrete examples and modeling, Reminders and development around the linear model (modeling, hypotheses, least squares and likelihood, Fisher/Student test, etc.)
  - Introduction to model selection (construction of Cp/AIC/BIC criteria, oracle inequalities, high-dimensional behavior)
  - Ridge method (heuristic, link with Tikhonov, property of risk)
  - Introduction to the LASSO method (construction and heuristics, link with compressed sensing, theoretical properties / oracle inequalities, compatibility conditions)
- Supervised Classification: Concrete examples and modeling, overview of some algorithms (kNN, SVM, neural networks, logistic regression, etc.), theoretical aspects (concentration inequalities, kernels, etc.)
- Unsupervised classification: PCA, Clustering (kmeans, hierarchical methods, etc.), Gaussian mixture models, Spectral clustering.
- Other risks, extremes, introduction to research problematics.

3.2 Second semester

P3 : Stochastic Partial Differential Equations and application to turbulence theory (Igor Honoré and Laurent Chevillard, 18h)

The aim of this course is to give an introduction to the stochastic partial differential which can be regarded as an infinite dimensional stochastic equation. The random noise, typically a white noise, appearing in many physical models, may imply some difficulties in the definition of the solution (*e.g.* Kardar-Parisi-Zhang equation); or, in some cases, a regularisation by noise may occur and well-posedness can be restored. This theory strongly relies on stochastic calculus and on PDE techniques.

The course is organised in two parts. The first one deals with the fundamental notions. The theory of probability on Banach and Hilbert spaces are explained, in particular the construction of the Brownian and the stochastic process in infinite dimension. Some representations of a SPDE solution are presented (via semi-group and variational formula), existence and uniqueness are also established in suitable cases. Some existing results and tough questions (due to the lack of regularity) are highlighted in some non-linear cases. Several probabilistic properties are deduced: Markovian, ergodic process...

In a second part, shorter than the first one, we propose an introduction to fluid turbulence, and the phenomenology developed by Kolmogorov and others. Fluid turbulence can be seen as the study of the solution of the Navier-Stokes equations, forced by a smooth in space random term. We will illustrate several surprising experimental facts about the statistical structure of the velocity field, and its fractal nature, with some ingredients of the equations of motion, and formulate some possible, yet to be done, mathematical developments.

References:

1. Cerrai, S. *Second order PDE's in finite and infinite dimension*, vol. 1762 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001. A probabilistic approach.

2. Da Prato, G., and Zabczyk, J. *Stochastic equations in infinite dimensions*, vol. 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.

3. Da Prato, G., and Zabczyk, J. *Ergodicity for infinite-dimensional systems*, vol. 229 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996.

4. Flandoli, F. *Random perturbation of PDEs and fluid dynamic models*, vol. 2015 of Lecture Notes in Mathematics. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.

5. Flandoli, F., Gubinelli, M., and Priola, E. *Well-posedness of the transport equation by stochastic perturbation.* Invent. Math. 180, 1 (2010), 1–53.

6. Frisch, U. *Turbulence: the legacy of AN Kolmogorov.* Cambridge university press, 1995.

7. Prévôt., C., and Röckner., M. *A concise course on stochastic partial differential equations*, vol. 1905 of Lecture Notes in Mathematics. Springer, Berlin, 2007.

P4 : Introduction to interacting particle systems (Clément Erignoux, 18h)

Interacting particle systems are a broad class of statistical physics models with many components, evolving in time as Markov processes. The course will focus on lattice gases in which space evolution occurs on a discrete lattice. Such systems can model a broad range of physical behavior, from glassy phase transitions, to epidemics, coagulation and condensation phenomena, heat conduction and non-equilibrium behavior. In this brief introduction, I will present the most classical interacting particle systems, both in the non-conservative (Glauber type) and conservative (Kawasaki) settings. I will present some classical questions that have been investigated on these models, as well as classical tools to tackle those.

P5 : Branching random walks on $\mathbb{Z}^d$ (Bruno Schapira, 18h)

In recent years branching random walks have become a central topic in probability theory. We will review some recent advances in the development of a potential theory for these processes, in analogy with what is known for usual random walks; in particular we will introduce the notion of branching capacity, infinite shift-invariant random tree, Green's function, etc..If time permits we will also discuss some links with percolation theory in high dimension.

P6 : Large Random Matrices, Large Deviations, and Applications (Alice Guionnet, 18h)

Large random matrices were introduced into statistics by Wishart over a century ago, then studied by Wigner and Dyson for their applications in nuclear physics. Since then, they have become a natural mathematical object in many areas of mathematics, statistics, and physics. This course will focus more specifically on the study of their large deviations and some applications. No prerequisites in large random matrices or large deviations are required.

S3 : Graphs and ecological networks (Clément Marteau and Thibault Espinasse, 18h)

[Also in the M2 *Math en Action* program]

A graph, whose first use are mentionned in the 16th century, is a mathematical object widely used from the first appearance of network investigations, namely investigation of relationship between individual in wide sense. Ranging from social network to the internet, graphs are leading objects for the analysis of several data sets. Ecosystem relationships, from species relationship (prédation, interaction between plants and pollinating insects , etc...) social relationship between individuals (sociality between primates, etc...), offers several different possible applications of graphs modelling and network investigation.

In this course, we will investigate the framework of graph theory and network science. We will provide an introduction to modern research problems regarding ecosystems studies. We will use alternatively discrete mathematics, statistics and machine learning.

We will adress both theoretical and practical (case studies in ecology) questions.

Theretical keywords: Bases / definitions (graphs, path, etc...) ; Metrics ; Clustering methods ; Spectral methods ; Random graphs models ; Graphical models (graphs inference) ; Signal processing on graphs ; Multi-level graphs (time, space, link types) ; Embedding methods (optional)

Case studies : Contact network between animals. Interaction network between species in a marine and/or alpine environment. Consideration about the relevance of a graph for biodiversity support.

S4: Neural networks (Cédric Gerbelot, Aurélien Garivier, Rémi Gribonval, Julian Tachella, 18h)

[Also in the M2 *Math en Action* program]

The goal of this course is twofold:

- To present the principles of modern deep neural networks, as well as the technical ways to implement them for solving classification and regression problems.
- To provide a detailed overview of the mathematical foundations of modern learning techniques based on deep neural networks.

Starting with the universal approximation property of neural networks, we will then see why depth improves the capacity of networks to provide accurate function approximations for a given computational budget. Tools to address the optimization problems appearing when training networks on large collections will then be covered, and their convergence properties will be reviewed. Finally, statistical results on the generalization guarantees of deep neural networks will be presented, both in the classical underfitting scenario and in the overfitting scenario leading to the so-called "double descent" phenomenon.

S5: Optimal transport and learning (Filippo Santambrigio, Ivan Gentil, Yohann de Castro, Julie Digne, Nicolas Bonnel, 18h)

[Also in the M2 *Math en Action* program]

The aim of the course is to present the broad outlines of optimal transport theory and some of its applications in data sciences.

A first part of the course will detail the Monge-Kantorovich problem, its formulation as a linear programming problem and the use of convex duality, as well as the distances (called Wasserstein distances) that optimal transport makes it possible to define in space. probability measures. Geodesics and barycenters in Wasserstein space, of great importance in the interpolation and comparison of data, will also be introduced.

A second part of the course will focus on numerical methods for solving optimal transport problems, with particular attention to the methods best suited to high dimension and unstructured data, in particular the Sinkhorn algorithm.

Finally, the third part of the course will present a choice of applications of transport and Wasserstein distances in learning, of which we cite as examples Wasserstein GANs, transfer learning, data generation models, etc.

S6: Inverse problems and parsimony (Yohann de Castro and Rémi Gribonval, 18h)

[Also in the M2 *Math en Action* program]

Sparsity and convexity are ubiquitous notions in Machine Learning and Statistics. In this course, we study the mathematical foundations of some powerful methods based on convex relaxation: L1-regularisation techniques in Statistics and Signal Processing; Nuclear Norm minimization in Matrix Completion; K-means and Graph Clustering.

These approaches turn out to be Semi-Definite representable (SDP) and hence tractable in practice. The theoretical part of the course will focus on the performance guarantees for theses approaches and for the corresponding algorithms under the sparsity assumption. The practical part of this course will present the standard SDP solvers for these learning problems.

Keywords: L1-regularisation; Matrix Completion; K-Means; Graph Clustering; Semi-Definite Programming.