MA2 2021-2022

Partial Differential Equations and Applications

This program aims to prepare students for research in the field of theoretical and numerical analysis of problems involving partial differential equations (PDEs). It has three components:

1. Refresher Courses in the first 2.5 weeks aimed at ensuring a common knowledge base for students from various mathematical backgrounds. These courses are optional but very strongly advised.

2. Three Basic Courses which offer a broad introduction to the analysis techniques of a large class of partial differential equations.

3. Four Advanced Courses on subjects closely related to current research: the analysis of equations with stochastic components, the mean-field problem in quantum mechanics, the optimal transport theory for parabolic equations, and numerical methods for approximation of PDEs.

The advanced courses will particularly welcome the participation of PhD students and colleagues.

1 Refresher Courses

Basic tools of functional analysis, Simon Masnou (16h)

1. Duality: Hahn-Banach theorem, weak and weak-* topologies, Lebesgue spaces;
2. Distributions: weak derivatives, convolution, fundamental solutions of differential operators;
3. Fourier transform;
4. Sobolev spaces: embeddings, extension and traces, compactness;
5. Weak solutions of PDEs;
6. Spectral analysis in Hilbert spaces.

Stochastic tools, Grégory Miermont (15h)

1. Discrete time martingales: stopping theorems and convergence. Extensions for continuous time martingales.
2. Construction of Brownian motion. Regularity of trajectories.
3. Some properties of Brownian trajectories. Connection with the heat equation.

Starting with PDEs, Francesco Fanelli (16h)

1. Introduction: classifications of PDEs, symbols, notions of solutions.
2. The Laplace equation and second order elliptic operators.
3. The heat equation and second order parabolic operators.
4. Hyperbolic operators.
5. Semigroup theory and applications.

2 Basic Courses

Evolution equations, Emmanuel Grenier (24h)

The aim of these lectures is to study some of the following evolution equations:

1. reaction-diffusion
2. hyperbolic systems of conservation laws
3. parabolic equations
4. simple kinetic equations
5. Euler and Navier-Stokes equations.

We will investigate existence, uniqueness, smoothness and qualitative properties of the corresponding solutions.

Calculus of variations and elliptic equations, Filippo Santambrogio (24h)

The course will be mainly devoted to the study of the minimizers of integral functionals, their existence, their regularity, and their characterization in terms of solutions of some partial differential equations, but regularity results for the equations themselves will also be presented for their own interest.
The course will be roughly structured into 10 classes as follows:

1. *Introduction and 1D* examples of 1D variational problems (geodesics, brachistochrone, economical growth models) and their applications, tools for existence, Euler-Lagrange equation (both in 1D and in higher dimension).
2. *Convexity and semicontinuity* conditions to ensure the semicontinuity for the weak Sobolev convergence of integral functionals and applications to existence results. Notions of convex analysis (Fenchel-Legendre transforms, subdifferentials. . .).
3. *Convex duality* duality for some "simple" convex variational problems.
4. *Regularity via duality* application of convex duality to some H^1 regularity results.
5. *Harmonic functions and distributions* main properties of the solutions of $\Delta u = 0$.
6. L^p *estimates for the Poisson equation.* Proof by interpolation of the result $\Delta u = f$, $f \in L^p \Rightarrow u \in W^{2,p}$.

7. *Hölder regularity with smooth coefficients.* Morrey-Campanato spaces and applications to the result $\nabla\cdot(a(x)\nabla u) = \nabla\cdot F$, $a, F \in C^{k,\alpha} \Rightarrow u \in C^{k+1,\alpha}$.

8. *Hölder regularity with bounded coefficients.* Proof by Moser's iterations of the De Giorgi regularity result $\nabla\cdot(a(x)\nabla u) = 0$, a bounded and uniformly elliptic but not smooth $\Rightarrow u \in C^{0,\alpha}$ and applications to the solution of the 19th Hilbert problem.

9. $\Gamma-$*convergence and examples.* The general theory of the $\Gamma-$convergence for the limits of variational problems and some example, in particular the optimal quantization of measures (aka optimal location problem).

10. *BV functions, perimeters, and the Modica-Mortola functional.* Few words about the space BV and its role in defining sets of finite perimeter. Proof of the $\Gamma-$convergence of the functionals $\int \varepsilon|\nabla u|^2 + \varepsilon^{-1}W(u)$ towards the perimeter functional.

A detailed bibliography and a list of exercises will be provided.

The knowledge of some functional analysis (in particular, compactness for weak-* convergence and Sobolev spaces) and some measure theory is the main prerequisite for the course.

Discontinuous finite-element methods and applications , Daniel Le Roux (24h)

The goal of the proposed course is to describe the Discontinuous Galerkin methods and to discuss their main features and applications. We concentrate on the exposition of the ideas behind the devising of these methods as well as on the mechanisms that allow them to perform so well in a variety of problems: hyperbolic, elliptic and parabolic. Completely discontinuous approximations are highly parallelizable, they easily handle irregular meshes with hanging nodes and approximations that have polynomials of different degrees in different elements. Moreover, the methods are locally conservative, stable, and high-order accurate. Finally, when applied to non-linear hyperbolic problems, the discontinuous Galerkin methods are able to capture highly complex solutions presenting discontinuities with high resolution.

The course is intented to be divided in four parts.

- The first part is dedicated to the continuous finite-element method. We start by explaining what mixed methods are. To this aim, by employing the Stokes problem for viscous incompressible flow as an example, we write down the corresponding minimization problem and thanks to the duality methods we come up with a saddle point problem. A few results for mixed methods are then presented for both the abstract and discrete settings: existence, uniqueness and stability of the solution and error estimates. Stabilization procedures (SUPG, GLS etc. methods) aiming to prevent the appearence of spurious solutions at the discrete level are finally presented. This makes the link with the discontinuous Galerkin methods, which may be regarded somehow as stabilized finite-element methods.

- In the second part, we introduce the Discontinuous Galerkin methods for hyperbolic problems. First, in 1D starting with the linear transport equation, and secondly with the linear 2D shallow-water system. In both cases, the discontinuous variational formulation is introduced with the appropriate numerical traces, and the stability of the solutions and error estimates are discussed. Fourier analyses complete the study by computing the dispersion relations at the discrete level for the frequencies, bringing additional informations about the stability and accuracy of the computed solutions. Numerical results illustrate the theoretical results

- In the third part, the linear 1D Poisson problem and 2D heat equation are examined by following the same methodology and approach than in the second part. We show how to rewrite second order problems in the context of discontinuous methods to ensure a well-posed problem. The choice of the numerical traces is now more delicate than in the case of hyberbolic problems. The LDG (Local Discontinuous Galerkin) method ensuring stability is chosen for the traces in the Fourier analysis.

- Finally, in the last part of the course, time discretization methods and non linear problems are investigated. In particular the non linear shallow water model is discretized and we show how to compute the numerical traces efficiently by using the PVM (Polynomial Viscosity Method). Numerical results of the propagation of Rossby waves for environmental flows illustrate the capability of the Dicontinuous Galerkin method to handle serious problems.

3 Advanced Courses

Stochastic PDEs and their asymptotic behaviour,

Alexandre Boritchev (18h)

This course, although part of the PDE program, could also interest students in probability theory; for more information, see the prerequisites listed at the end of the description.

The main technicalities come from the complex interplay between the regularity of the solution to the deterministic equation and that of the noise. We also give some key notions on long-term behaviour of solutions and on the role of the stationary measure. The latter is a probability measure on a functional space which is the random counterpart of a stationary solution for a deterministic equation.

This is a course which introduces many new concepts: therefore we will follow a plan which underlines the differences and similarities between the finite-dimensional situation (stochastic ODEs) and the infinite-dimensional one (stochastic PDEs).

We use heavily the material from all refresher courses: "Basic tools of functional analysis" (convolution, Sobolev spaces...), "Stochastic tools" (Brownian motion and regularity of its trajectories), "Starting with PDEs" (parabolic equations, semigroup formalism), as well as from the "Evolution equations" course.

1. Introduction:
   - Wiener process: a reminder. Regularity of the trajectories.
   - The Markov property: how does it work?
2. SDEs:
   - What is a stochastic differential equation (SDE)? Two points of view: ODEs with random coefficients and through the Itô integral.
   - The Markov semigroup on the space of probability measures.
   - Stationary measure and convergence to the equilibrium.
   - Example: the Ornstein-Uhlenbeck equation.
3. An introduction to SPDEs:
   - What is a stochastic partial differential equation (SPDE)?
   - The Markov semigroup on the space of probability measures in a Banach space.
   - Stationary measure and convergence to the equilibrium.
   - Examples: the stochastic heat and Burgers equations.

- More involved example (if time allows): the stochastic 2D incompressible Navier-Stokes system.

Optimal transport theory and links with parabolic equations, Ivan Gentil (18h)

Goal of the course : The goal of this course is to introduce the Wasserstein distance between two probability measures. This distance has many recent developments. For instance, this is a natural distance to show the asymptotic behavior of evolutional PDE, asymptotic behavior of stochastic process, to show concentration inequalities, and also to prove that the heat equation is the gradient flow of the Boltzmann entropy with respect to the Wasserstein metric.

We will see in this course analytic properties of the Wasserstein distance and also links between parabolic PDE and optimal transport. Parabolic PDE can be seen in this context as the law of diffusion Markov processes, solutions of SDE driven by a Brownian motion.

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1. Properties of the Wasserstein distance between two probabilities measures.
   (a) Optimal transportation, Monge and Monge-Kantorovich problems.
   (b) Properties of the Wasserstein space, in particular its geodesic property.
   (c) Duality of Kantorovich.
   (d) Brenier's Theorem and proof of the optimal Sobolev inequality by using the optimal transport.
2. Links between parabolic PDE and Otto calculus.
   (a) Notion of operator's curvature.
   (b) Von Renesse-Sturm's Theorem : equivalence between curvature of an operator and the contraction in Wasserstein distance.
   (c) Heat equation as a gradient flow of the entropy with respect to the Wasserstein metric, introduction of the Otto Calculus.

Some references

1. L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
2. D. Bakry, I. Gentil et M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Cham, 2014.

3. C. Villani – *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.

4. C. Villani *Optimal transport*, Grundlehren der Mathematischen Wissenschaften, vol. 338, Springer-Verlag, Berlin, 2009, Old and new.

Many-body quantum mechanics and mean-field limits,
Nicolas Rougerie (18h)

How and why could an interacting system of many particles be described as if all particles were independent and identically distributed ? This question is at least as old as statistical mechanics itself. Presupposing the answer, it leads to the mean-field approximation: particles are assumed to follow a single statistical law that interacts with itself via the mean interaction generated by the other particles.

In this course we shall study various mathematical techniques allowing to vindicate the validity of the mean-field approximation in a reasonable macroscopic limit of large particle number. We will focus on energy minimizers/ground states of the basic many-body Hamiltonian and prove that they do behave as if all particles were independent and identically distributed.

Topics we plan to cover include:

- Recap of basic spectral theory and functional analysis. Self-adjointness of a Schrödinger Hamiltonian.
- Review of many-body quantum mechanics. Symmetry types of quantum particles, bosons and fermions. Second quantized formalism.
- Study of mean-field models: non-linear Schrödinger equation (mostly static), Thomas-Fermi type models.
- The de Finetti-Hewitt-Savage theorem in statistical mechanics. Proof according to Diaconis and Freedman.
- Mean-field limits of classical equilibrium states.
- Basic tools in mathematical quantum mechanics: Onsager's lemma, Hoffmann-Ostenhof inequality, Lieb-Thirring and Lieb-Oxford inequality ...
- Coherent state formalism for large bosonic systems and quantum de Finetti theorem.
- Semi-classical limit of large fermionic systems.

The course will borrow from review papers/lecture notes available at:

https://arxiv.org/abs/2002.02678
https://arxiv.org/abs/1506.05263

The course will follow a different path however, proofs will be much more detailed and basic mathematical tools will be recaped more thoroughly.

Prerequisites : It is desirable, although not absolutely necessary, to have followed the course Calculus of variations and elliptic equations (by Filippo Santambrogio).

Numerical approximation methods for fluid mechanics, Khaled Saleh (18h)

A first objective of this course is to present recent methods for the numerical approximation of fluid mechanics models. The considered numerical schemes are used in industrial codes. They are designed so as to mimic the main properties satisfied by the exact solutions: positivity of the density, mass and momentum conservation, energy estimates, etc. A second objective of the course is to establish the convergence of the numerical schemes by following the lines of the theory of existence of weak solutions for the considered PDE models. For this purpose, famous functional analysis results must be adapted to discrete functional spaces.

Contents of the course:

1. PDE models in fluid mechanics: compressible Navier-Stokes and Euler systems. Incompressible models.
2. Study of the incompressible Stokes model. Well-posedness for weak solutions. Analysis of a finite element numerical scheme: a priori estimates, compactness is assumed at this level, convergence of the approximate solutions towards the exact weak solution.
3. Discrete functional analysis. We prove continuous/compact embedding results for discrete functional spaces arising from the numerical discretization. These are discrete counterparts to the famous Sobolev continuous embedding and Rellich's compact embedding theorems.
4. Numerical analysis of a finite volume - finite element numerical scheme for the compressible Navier-Stokes equations: positivity and energy estimates, convergence of the numerical method (compactness, passing to the limit in non-linear terms with only weak convergence) for the stationary model.
5. Non stationary models. Compactness in time: Aubin-Simon theorem and its discrete counterpart. Study of a numerical scheme for the non stationary Stokes model.