p-adic geometry and representations

1 Refresher course

Riccardo Pengo (to be determined)

2 Basic courses

Laurent Berger : Local fields

This course is an introduction to arithmetic and analysis in local fields, in particular p-adic fields. The following topics will be covered : local fields, Galois theory, ramification theory, p-adic analysis, formal groups, local class field theory and p-adic periods. Additional topics may be treated depending on the other courses' needs.

Olivier Taïbi : Modular and automorphic forms

This course is an introduction to the theory of modular and automorphic forms, focusing on the case of the group GL_2 over $\mathbb{Q}$. These objects lie at the intersection of algebraic geometry, analysis and representation theory, but they are mainly studied for their arithmetic aspect, in particular in the Langlands program which seeks to link them to ℓ-adic Galois representations and motives. The course will cover some of the following topics :

— classical modular forms, arithmetic applications ;
— adelic reformulation, representation theory ;
— representations of $\mathrm{GL}_2(\mathbb{R})$;
— spectral decomposition of cuspidal automorphic forms ;
— representations of $\mathrm{GL}_2(\mathbb{Q}_p)$;
— examples for groups other than GL_2 ;
— Selberg trace formula ;
— Eisenstein series.

In any case the link with Galois representations will be stated without proof.

Sophie Morel : Introduction to rigid analytic geometry

This course is an introduction to rigid analytic varieties, from the classical point of view and (time permitting) the point of view of Berkovich spaces. We will talk about the following subjects : Tate algebras, affinoid algebras and affinoid spaces, Tate acyclicity, definition of rigid spaces, coherent sheaves, proper/finite/separated morphisms, flat morphisms.

Reading list (covering much more than what I will be able to do in class) :

— Bosch, *Lectures on formal and rigid geometry* ;
— Bosch, Güntzer, Remmert, *Non-Archimedean analysis* ;
— Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields.*

3 Advanced courses

Sandra Rozensztajn : Galois representations, deformations, and families of Galois representations.

The goal of this course in to introduce Galois representations and their deformations. We will start by the notion of linear representation of a global of local Galois group, and the properties of these representations. We will introduce deformations and the universal deformation rings attached to them. Finally we will introduce the "eigencurve", which can be seen as a universal family of Galois representations coming from finite slope modular forms. All these objects are important technical tools for many questions in number theory related to Galois representations and automorphic forms.

Amaury Thuillier : Rigid geometry of algebraic curves

It is well-known that an algebraic curve defined over the field of complex numbers can be seen as a Riemann surface and, as such, studied by complex analytic tools. Over a field k complete with respect to a non-Archimedean absolute value, a similar picture emerged around 1960, when Tate laid the foundations of rigid geometry. The main purpose of this course is to illustrate the general theory developped in Sophie Morel's lectures by looking closely at curves, and to present some applications. A side goal is to introduce Berkovich's approach to non-Archimedean analytic geometry, which highlights the nice topology hidden in classical rigid geometry.

The first part of the course will focus on the analytic structure of an algebraic curve X over a non-Archimedean field k. Whereas a smooth algebraic curve defined over the field of complex numbers is locally isomorphic to a disk from the analytic point of view, the situation is more subtle in the non-Archimedean setting. In fact, the analytic structure of X reflects how this curve "degenerates" over the residue field of k, i.e. the existence of a model of X over the ring of integers in k with only mild singularities over $\tilde{k}$ (*Semi-stable reduction Theorem*).

In the second part, we will study non-Archimedean uniformization of curves, which parallels Schottky uniformization of Riemann surfaces. Schottky groups are special subgroups Γ of $\mathrm{PGL}_2(k)$ acting nicely on some open domains Ω of the projective line, and a smooth projective curve X over k can be written Ω/Γ in the analytic category, provided X is locally isomorphic to $\mathbf{P}^1_k$ in this category.

The last part of the course will present some applications of rigid geometry of curves to inverse Galois theory (Harbater's formal patching) and the search of rational points of curves over number fields (Chabauty-Coleman method).

References

[1] V. Berkovich, *Spectral theory and analytic geometry over non archimedean fields*, Mathematical Surveys and Monographs 33, AMS, 1990.

[2] D. Harbater, *Patching and Galois theory*, in *Galois Groups and Fundamental Groups*, Cambridge University Press, 2003.

[3] W. Lütkebohmert, *Rigid Geometry of curves and their Jacobians*, volume 61 of *Ergebnisse der Mathematik und ihre Grenzegebiete*, Springer, 2016.

[4] W. McCallum and B. Poonen, *The method of Chabauty and Coleman*, in *Explicit methods in number theory ; rational points and diophantine equations*, SMF, 2012

[5.] J. Poineau and D. Turchetti, *Berkovich curves and Schottky uniformization*, `arXiv:2010.09043` (2020)

Bertrand Rémy : Reductive groups over local fields (Bruhat-Tits theory)

The aim of the course is to provide an introduction to the theory of semissimple algebraic groups (e.g. SL(n) or almost any other classical matrix group) over local fields (e.g. finite extensions of the p-adic completion of the field of rational lumbers). The first part will consist in introducing linear algebraic groups. The second part will present the theory that makes it possible to understand the rational points of these groups : this is done by making them act on well-adapted cellular metric spaces, the affine buildings.