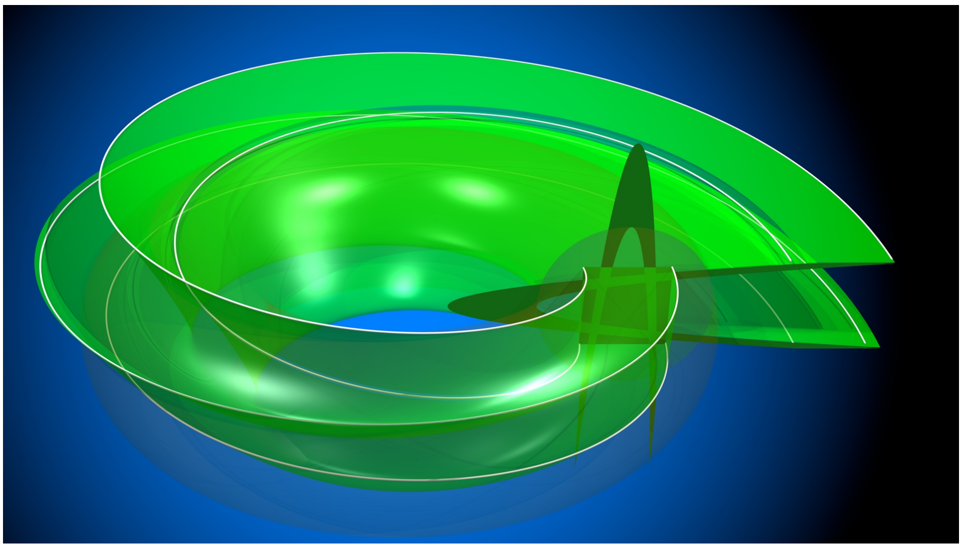
GEOMETRY AND DYNAMICS

MA2 2022-2023

This program aims to prepare students for research in geometry and dynamics with an emphasize in complex geometry. In the two first weeks, refresher courses (30h) will be proposed in order to ensure a common knowledge base for students from various mathematical backgrounds. After that, three basic courses of 24 hours each will be given during the first semester, before three more advanced courses during the second part of the year.

Contents



Smale horseshoe (https://www.chaos-math.org)

Refresher courses

0.1. Riemann surfaces. (6h) Aurélien Alvarez

0.2. Commutative algebra. (6h) Stéphane Druel

0.3. Functions of several complex variables. (6h) Jean-Claude Sikorav

0.4. Topics in Riemannian geometry. (12h) Jean-Claude Sikorav and Ghani Zeghib

Semester 1

1.1. Complex algebraic geometry. (Sophie Morel)

This course is an introduction to the theory of complex algebraic varieties. Here is a list of topics that will hopefully be covered (the order of the topics may change):

- Definition of complex algebraic varieties and morphisms between them; affine, projective and quasi-projective varieties.
- Dimension, Noether normalization theorem.
- Constructible subsets and Chevalley's theorem.
- Separated and proper morphisms, complete varieties.
- Finite and affine morphisms.
- Tangent space, smooth varieties and smooth morphisms.
- Quasi-coherent sheaves and their cohomology, Serre's theorems A and B.
- Invertible sheaves and divisors, Riemann-Roch theorem.
- Birational morphisms and blowups.
- (Time permitting) Intersection theory on surfaces.

References

- D. Perrin, *Géométrie algébrique. Une introduction* (also exists in English translation).
- Shafarevich, *Basic algebraic geometry I.*

1.2. Basic course on differentiable dynamical systems. (Jean-Claude Sikorav)

This course is an introduction to the global study of differentiable dynamical systems on manifolds i.e., the iteration of a diffeomorphism or the flow of a vector field. We shall mostly investigate the following two questions:

- finding criteria for such a system to be structurally stable, meaning that any "close enough" system is topologically equivalent to it; the most important tool to answer it is that of hyperbolicity of various closed invariant subsets: fixed or periodic points, periodic orbits, attractors of various types, and the whole space in the case of Anosov flows.
- find a "nice" description of structurally stable systems, usually in terms of "symbolic dynamics".

References

- A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Smooth Dynamical Systems*, Cambridge University Press, 1995.
- L. Wen, *Differentiable Dynamical Systems*, Graduate Studies in Math. 173, Amer. Math. Soc., 2016.

1.3. **Introduction to Lie groups and symmetric spaces.** (Bruno Sévennec)

This is a first semester course, introducing the notions of Lie group and Lie algebra (over the real or complex numbers), their homogeneous spaces, and the geometry of symmetric spaces. On this last subject, we will follow Donaldson's approach, who proves existence and "uniqueness" of a maximal compact subgroup in any simple Lie group, based on a minimizing argument in the symmetric space $\mathrm{SL}_n/\mathrm{SO}_n$.

Tentative list of subjects to be taught:

- General facts about Lie groups and Lie algebras (adjoint action, exponential, morphisms, Baker-Campbell-Hausdorff formula...)
- Simple and semi-simple Lie algebras (real vs. complex).
- Actions on manifolds, homogeneous spaces, symmetric spaces.
- Maximal compact subgroups of adjoint simple Lie groups, after Donaldson.
- Compact semi-simple groups, maximal tori, root systems.

Semester 2

2.1. **Introduction to symplectic dynamics.** (Marco Mazzucchelli)

Hamiltonian systems are an important class of dynamical systems, that allow for instance to describe the planetary motions, the geodesics of Riemannian manifolds, the trajectories of convex billiards, and so forth. In this course, we will present some classical and more modern symplectic techniques for the study of Hamiltonian systems, and in particular of their periodic orbits.

The first part of the course will be devoted to introducing symplectic and contact manifolds, Hamiltonian flows and diffeomorphisms, Reeb flows, and some of the tools that allow to study them: generating functions, holomorphic curves, symplectic capacities, and symplectic homology.

The final part of the course will be an introduction to some of the celebrated results and open problems in the field: in particular the Arnold conjecture, the Weinstein conjecture, the Viterbo conjecture, as well as others.

Pre-requirements: basic notions from differential geometry (manifolds, vector fields, differential forms), functional analysis, and possibly some algebraic topology (fundamental group, homology, cohomology).

References

- M. Audin, M. Damian, *Morse theory and Floer homology*, Springer.
- D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford University Press.
- H. Geiges, *Introduction to Contact Topology*, Cambridge University Press.
- H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser.

2.2. **Holomorphic foliations.** (Stéphane Druel)

The course will give an introduction to the study of (possibly singular) holomorphic foliations on complex projective surfaces. We will present several aspects of their birational geometry. Topics that will be covered include:

- Foliations (definition and first properties).
- Local theory.
- Examples.
- Index theorems.
- Algebraic integrability.
- Minimal models.

References

- M. Brunella, *Birational geometry of foliations*, IMPA Mathematical Publications, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2004.
- F. Cano, D. Cerveau, J. Déserti, *Théorie élémentaire des feuilletages holomorphes singuliers*, Belin, 2013.

2.3. **Dynamics of complex differential equations.** (Aurélien Alvarez - Ghani Zeghib)

The study of differential equations in the complex domain is a relatively old subject which dates back to the end of the 19th century but continues to arouse great interest. These are differentiable equations with a complex variable and algebraic coefficients. After compactification, we come back to the study of algebraic (singular) foliations on $\mathrm{P}^2_{\mathbf{C}}$. The aim of the course is to study the dynamics and the geometry of certain classes of such foliations. The main ingredient is monodromy which is a linear representation of the fundamental group of the complement of the set of singularities. Some of the topics that will be covered in this course include

- linear case: Fuchsian equations, Fuchsian groups, Riemann-Hilbert problem (monodromy inverse problem);
- the differential equations of Riccati and their monodromies;
- the equations of Painlevé and their applications in various mathematical fields;
- the algebraic structure of the foliation space of the projective plane $\mathrm{P}^2_{\mathbf{C}}$ and (time permitting) the structural stability of the equation of Jouanolou.

References

- N. Bergeron, *Differential equations with algebraic coefficients over arithmetic manifolds.* The scientific legacy of Poincaré, 47–71, Hist. Math., 36, Amer. Math. Soc., Providence, RI, 2010.
- F. Beukers, Notes on differential equations and hypergeometric functions.
- F. Cano, D. Cerveau, J. Déserti, *Théorie élémentaire des feuilletages holomorphes singuliers*, Belin, 2013.
- H. P. de Saint-Gervais, *Uniformisation des surfaces de Riemann*, ENS Éditions, 2010.
- R. Tazzioli Fonctions fuchsiennes ou schwarziennes? Mieux poincaréennes, *Images des mathématiques*, 2010.