MA2 2022-2023

Partial Differential Equations and Applications

This program aims to prepare students for research in the field of theoretical and numerical analysis of problems involving partial differential equations (PDEs). It has three components:

1. Refresher Courses in the first 2.5 weeks aimed at ensuring a common knowledge base for students from various mathematical backgrounds. These courses are optional but very strongly advised.

2. Three Basic Courses which offer a broad introduction to the analysis techniques of a large class of partial differential equations.

3. Four Advanced Courses on subjects closely related to current research: optimal transport, the Navier-Stokes equation, the mean-field theory in quantum mechanics, and numerical methods for fluid equations.

The advanced courses will particularly welcome the participation of PhD students and colleagues.

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1 Refresher Courses

Basic tools of functional analysis, Mikael De La Salle (16h)

1. Duality: Hahn-Banach theorem, weak and weak-* topologies, Lebesgue spaces;
2. Distributions: weak derivatives, convolution, fundamental solutions of differential operators;
3. Fourier transform;
4. Sobolev spaces: embeddings, extension and traces, compactness;
5. Weak solutions of PDEs;
6. Spectral analysis in Hilbert spaces.

Starting with PDEs, Francesco Fanelli (16h)

1. Introduction: classifications of PDEs, symbols, notions of solutions.
2. The Laplace equation and second order elliptic operators.
3. The heat equation and second order parabolic operators.
4. Hyperbolic operators.
5. Semigroup theory and applications.

Stochastic tools, ? (15h)

1. Discrete time martingales: stopping theorems and convergence. Extensions for continuous time martingales.
2. Construction of Brownian motion. Regularity of trajectories.
3. Some properties of Brownian trajectories. Connection with the heat equation.

2 Basic Courses

Evolutionary PDEs, Dragos Iftimie (24h)

1. Some properties and reminders of distributions.
2. The Cauchy problem for linear PDEs.
   (a) Variable coefficients. Cauchy-Kovalevskaya theorem, characteristic hypersurfaces and Holmgren's uniqueness theorem. Well-posed problems.
   (b) Constant coefficients.
      - Existence of an elementary solution, the Malgrange-Ehrenpreis theorem. Examples. Necessary and sufficient conditions for hypoellipticity.
      - Local resolubility of the Cauchy problem. Hyperbolicity. Gårding's theorem. Necessary and sufficient conditions for hyperbolicity.
3. Dispersive PDEs.
   (a) A few linear dispersive PDEs and their explicit solutions.
   (b) Non linear Schrödinger equation. Strichartz estimates and some well-posedness results for the Cauchy problem.
4. Symmetric hyperbolic quasilinear systems. Incompressible Euler equations. H^3 solutions and the Beale-Kato-Majda blow-up criterion.
5. Incompressible Navier-Stokes equations. Leray solutions. Uniqueness for small data in dimension 3.

Discontinuous finite-element methods and applications, Daniel Le Roux (24h)

The goal of the proposed course is to describe the Discontinuous Galerkin methods and to discuss their main features and applications. We concentrate on the exposition of the ideas behind the devising of these methods as well as on the mechanisms that allow them to perform so well in a variety of problems: hyperbolic, elliptic and parabolic. Completely discontinuous approximations are highly parallelizable, they easily handle irregular meshes with hanging nodes and approximations that have polynomials of different degrees in different elements. Moreover, the methods are locally conservative, stable, and high-order accurate. Finally, when applied to non-linear hyperbolic problems, the discontinuous Galerkin methods are able to capture highly complex solutions presenting discontinuities with high resolution.

The course is intented to be divided in four parts.

- The first part is dedicated to the continuous finite-element method. We start by explaining what mixed methods are. To this aim, by employing the Stokes problem for viscous incompressible flow as an example, we write down the corresponding minimization problem and thanks to the duality methods we come up with a saddle point problem. A few results for mixed methods are then presented for both the abstract and discrete settings: existence, uniqueness and stability of the solution and error estimates. Stabilization procedures (SUPG, GLS etc. methods) aiming to prevent the appearence of spurious solutions at the discrete level are finally presented. This makes the link with the discontinuous Galerkin methods, which may be regarded somehow as stabilized finite-element methods.

- In the second part, we introduce the Discontinuous Galerkin methods for hyperbolic problems. First, in 1D starting with the linear transport equation, and secondly with the linear 2D shallow-water system. In both cases, the discontinuous variational formulation is introduced with the appropriate numerical traces, and the stability of the solutions and error estimates are discussed. Fourier analyses complete the study by computing the dispersion relations at the discrete level for the frequencies, bringing additional informations about the stability and accuracy of the computed solutions. Numerical results illustrate the theoretical results

- In the third part, the linear 1D Poisson problem and 2D heat equation are examined by following the same methodology and approach than in the second part. We show how to rewrite second order problems in the context of discontinuous methods to ensure a well-posed problem. The choice of the numerical traces is now more delicate than in the case of hyberbolic problems. The LDG (Local Discontinuous Galerkin) method ensuring stability is chosen for the traces in the Fourier analysis.

- Finally, in the last part of the course, time discretization methods and non linear problems are investigated. In particular the non linear shallow water model is discretized and we show how to compute the numerical traces efficiently by using the PVM (Polynomial Viscosity Method). Numerical results of the propagation of Rossby waves for environmental flows illustrate the capability of the Dicontinuous Galerkin method to handle serious problems.

Calculus of variations and elliptic partial differential equations and systems, Petru Mironescu (24h)

Description. This is an intermediate + course presenting some basic tools in the qualitative analysis, existence, and regularity theory for solutions of elliptic partial differential equations (PDEs). A first part, related to the direct method in the calculus of variations, goes beyond elliptic PDEs.

Prerequisites

1. Good knowledge of general measure theory and integration.

2. Reasonable knowledge of geometric aspects of the integration theory (Gauss-Ostrogradskii...) and of the local theory of submanifolds of $\mathbb{R}^n$.

3. Good knowledge of the basic results concerning the Laplace equation.

Syllabus

1. The direct method in the calculus of variations
   (a) Basic examples.
   (b) Notions of convexity.
   (c) Passing to the weak limits in nonlinear quantities. Compensation phenomena.
   (d) Gap phenomena.

2. Maximum principles and applications
   (a) Maximum principles for elliptic partial differential equations (PDEs) in non divergence and divergence form.
   (b) Iterative methods based on monotonicity (sub- and supersolutions).
   (c) Symmetry properties of solutions of semilinear elliptic PDEs.
   (d) Krasnoselskii's uniqueness result.

3. Regularity theory
   (a) Serrin's example.
   (b) Singular integrals.
   (c) L^p-theory for elliptic equations in non-divergence form.
   (d) A glimpse of the C^α-theory for elliptic equations in non-divergence form.
   (e) De Giorgi-Nash regularity theory for elliptic equations in divergence form.
   (f) Bootstrap. Regularity in the critical case.
   (g) Equations with L^1 or measure right-hand side.
   (h) A limiting case: Wente estimates. A glimpse of other compensation phenomena.

4. Other existence methods
   (a) Continuation methods for semilinear elliptic PDEs.
   (b) Crandall-Rabinowitz bifurcation from simple eigenvalues.
   (c) Concentration-compactness (sketch).
   (d) A glimpse of the Lyapunov-Schmidt reduction.
   (e) Mountain pass solutions. A glimpse of other topological methods.

5. Vector-valued problems
   (a) Hélein's two-dimensional regularity theorem ($\mathbb{S}^2$-case).

    (b) A glimpse of Rivière's three-dimensional example.
    (c) A glimpse of the Schoen-Uhlenbeck (Hardt-Lin) regularity theory for harmonic maps, with a focus on the salient points: clearing out (δ-regularity) and Federer's dimensional reduction argument.
    (d) Notions of weak solutions: kinetic formulation, entropies.

6. A glimpse of phase-transition problems
    (a) A glimpse of the BV space.
    (b) Abstract Γ-convergence.
    (c) The Modica-Mortola functional in limit $\varepsilon \to 0$.
    (d) Vector-valued variants. A glimpse of the (simplified) Ginzburg-Landau theories.

3 Advanced Courses

Optimal transport: introduction, applications and derivation, Aymeric Baradat (18h)

The starting point of this course will be to introduce the optimal transport problem between two probability measures on a physical space (such as subsets of Euclidean spaces) both in a static and in a dynamical version. We will see that the dynamical one gives a way to interpret formally the set of probability measures as an infinite dimensional Riemannian manifold, and we will discuss the corresponding notion of gradient flows as well as some properties of the geodesics. We will also see how the induced distance (the so-called Wasserstein distance) can be used to get contraction estimates in some nonlinear PDEs of Vlasov type.

In a last part, we will show how to recover optimal transport as the limit when the diffusivity tends to zero of the problem of minimizing the relative entropy w.r.t. the law of a Brownian motion under marginal constraints: this is the so-called entropic regularization of optimal transport. In addition to building a bridge between the theories of optimal transport and of large deviations of stochastic processes, this approach provides a way to compute efficiently approximated solutions thanks to the Sinkhorn algorithm.

The first two parts of the lecture will demand very few prerequisites except from notions of measure theory and basic knowledge in functional analysis. For the last part, it is better to be familiar with stochastic processes, and in particular with the Brownian motion.

The outline of the course will be as follows.

1. The static optimal transport problem
    - The Monge formulation and its relaxation, the Monge-Kantorovich problem
    - The Brenier theorem
    - The Wasserstein distance between probability measures
2. A dynamic reformulation: the Benamou-Brenier approach
    - The theory of the continuity equation
    - The Benamou-Brenier formulation of optimal transport
    - A formal introduction to gradient flows
    - Geodesic convexity and applications
    - The Wasserstein distance for nonlinear PDEs of Vlasov type
3. Entropy minimization w.r.t. the law of a Brownian motion
    - Statement of the Schrödinger problem
    - Static and Benamou-Brenier reformulation
    - The Sinkhorn algorithm

- Γ-convergence towards optimal transport

I will provide a detailed bibliography during the lecture.

Compressible Viscous Flows with Low or Intermediate Regularity,
Didier Bresch (18h)

Motivation: The Navier-Stokes equations provide a basic mathematical model for describing the motion of a fluid. In the famous paper published in Acta Mathematica in 1934, "Sur le mouvement d'un fluide visqueux remplissant l'espace", Jean Leray (1906-1998) introduced the concept of weak solutions (and to do this, he also defined what is now called a Sobolev space), by giving a precise definition of what an irregular solution of the system is, and showed that there is such a weak solution for the homogeneous incompressible Navier-Stokes equations (see the Evolutionary PDEs basic course by D. Iftimie) .

We now call these solutions of minimal regularity (finite energy): solutions à la Leray. Even if the global existence of weak solutions does little about the well-posed character of the system, such an analysis has many practical interests and may help for intermediate regularity purposes. In addition to the physical meaning, because the regularity of the initial data assumed is minimal and strongly related to well-identified physical quantities, the stability properties of weak solutions on the continuous model help to better understand how to properly build stable numerical schemes that more often do not preserve strong regularity estimates (see works by T. Gallouet, R. Herbin, J.-C. Latché, E. Feireisl, T.G. Karper, A. Novotny and K. Saleh for instance). An other important kind of solutions (Hoff solutions) is the one with intermediate regularity where we allow density profile jumps with more informations on the velocity field.

Goal of the Course: This course aims at presenting some recent results related to the viscous compressible flows with low or intermediate regularity. We will compare the two notions of solutions. We will see that depending on the situations, different mathematical techniques have to be developed turning around transport equation with rough velocity fields.

Topics we plan to cover include:

- Compressible Navier-Stokes equations with density dependent viscosities: Nonlinear hypocoercivity properties and energy-entropy weak solutions.
- Compressible Navier-Stokes equations with non-monotone pressure laws: Appropriate weights in a non-local compactness tool.
- Compressible Navier-Stokes equations with an anisotropic viscous tensor: Hoff solutions with intermediate regularity in an L^p framework.

We will explain the physical motivations before describing some mathematical tools on simplified systems. It could be nice to know the introductory book by L.C. Evans on compactness: Weak convergence methods for nonlinear partial differential equations. Volume 74, CBMS Regional Conference series in Math, 1990. I will provide a detailed bibliography during the lecture (see also the Refresher Course by M. De La Salle).

Many-body quantum mechanics and mean-field limits, Nicolas Rougerie (18h)

How and why could an interacting system of many particles be described as if all particles were independent and identically distributed ? This question is at least as old as statistical mechanics itself. Presupposing the answer, it leads to the mean-field approximation: particles are assumed to follow a single statistical law that interacts with itself via the mean interaction generated by the other particles.

In this course we shall study various mathematical techniques allowing to vindicate the validity of the mean-field approximation in a reasonable macroscopic limit of large particle number. We will focus on energy minimizers/ground states of the basic many-body Hamiltonian and prove that they do behave as if all particles were independent and identically distributed.

Topics we plan to cover include:

- Recap of basic spectral theory and functional analysis. Self-adjointness of a Schrödinger Hamiltonian.
- Review of many-body quantum mechanics. Symmetry types of quantum particles, bosons and fermions. Second quantized formalism.
- Study of mean-field models: non-linear Schrödinger equation (mostly static), Thomas-Fermi type models.
- The de Finetti-Hewitt-Savage theorem in statistical mechanics. Proof according to Diaconis and Freedman.
- Mean-field limits of classical equilibrium states.
- Basic tools in mathematical quantum mechanics: Onsager's lemma, Hoffmann-Ostenhof inequality, Lieb-Thirring and Lieb-Oxford inequality ...
- Coherent state formalism for large bosonic systems and quantum de Finetti theorem.
- Semi-classical limit of large fermionic systems.

The course will borrow from review papers/lecture notes available at:

https://arxiv.org/abs/2002.02678
https://arxiv.org/abs/1506.05263

The course will follow a different path however, proofs will be much more detailed and basic mathematical tools will be recaped more thoroughly.

Prerequisites : It is desirable, although not absolutely necessary, to have followed the course "Calculus of variations and elliptic partial differential equations and systems" (by Petru Mironescu).

Numerical methods for yield stress fluids,
Paul Vigneaux (18h)

In this course, we will study numerical schemes for the resolution of the Navier-Stokes-Bingham equations. These equations describe the flow of materials which can be simultaneously in a fluid state or in a rigid state in the studied spatial domain, depending on the local stress of the material. Mathematically, the so called constitutive equation associated to yield stress fluids is discontinuous at the origin, contrary to the case of Newtonian fluids (standard Navier-Stokes equations, for e.g. water) where the law is linear.

Such materials appear in numerous contexts: industrial flows such as oil extraction and transport, food processing, civil engineering; geophysical flows such as landslides, debris flows or dense snow avalanches, *etc.* and some occurrences of biological flows.

Outline of the course:

- Mathematical modeling for yield stress fluid flows. Overview of key physical features and their links with the mathematical difficulties
- Reminders on numerical schemes for (Navier-)Stokes equations (links will be made with the course of D. Le Roux)
- Overview of numerical schemes for viscoplastic flows: pros and cos.
- Duality methods for yield stress fluid flows
- Associated discretizations in space
- Extensions

Introductory bibliography:

- N. J. Balmforth, I. A. Frigaard, and G. Ovarlez, ‘Yielding to Stress: Recent Developments in Viscoplastic Fluid Mechanics’, Annual Review of Fluid Mechanics, vol. 46, no. 1, pp. 121–146, 2014
- R. T. Rockafellar. Convex analysis. 1997
- G. Duvaut and J.-L. Lions. Inequalities in mechanics and physics. Springer-Verlag, 1976
- I. Ekeland and R. Temam. Convex analysis and variational problems. 1999.
- Beck and Teboulle. A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems. SIAM J. IMAGING SCIENCES. 2009
- P. Saramito, Complex fluids: modeling and algorithms, Springer, 2016.