MA2 2023-2024

Partial Differential Equations and Applications

This program aims to prepare students for research in the field of theoretical and numerical analysis of problems involving partial differential equations (PDEs). It has three components:

1. Refresher Courses in the first 2 weeks aimed at ensuring a common knowledge base for students from various mathematical backgrounds.
   These courses are optional but very strongly advised.

2. Three Basic Courses which offer a broad introduction to the analysis techniques of a large class of partial differential equations.

3. Four Advanced Courses on subjects closely related to current research: optimal transport, the Navier-Stokes equation, the mean-field theory in quantum mechanics, and numerical methods for fluid equations.

The advanced courses will particularly welcome the participation of PhD students and colleagues.

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1 Refresher Courses

Basic tools of functional analysis, Mikael De La Salle (16h)

1. Duality: Hahn-Banach theorem, weak and weak-* topologies, Lebesgue spaces;
2. Distributions: weak derivatives, convolution, fundamental solutions of differential operators;
3. Fourier transform;
4. Sobolev spaces: embeddings, extension and traces, compactness;
5. Weak solutions of PDEs;
6. Spectral analysis in Hilbert spaces.

Starting with PDEs, Alexandre Lanar (Boritchev) (16h)

1. Introduction: classifications of PDEs, symbols, notions of solutions.
2. The Laplace equation and second order elliptic operators.
3. The heat equation and second order parabolic operators.
4. Hyperbolic operators.
5. Semigroup theory and applications.

Stochastic tools, ? (15h)

1. Discrete time martingales: stopping theorems and convergence. Extensions for continuous time martingales.
2. Construction of Brownian motion. Regularity of trajectories.
3. Some properties of Brownian trajectories. Connection with the heat equation.

2 Basic Courses

PDE modeling in the natural sciences: an asymptotic viewpoint, Vincent Calvez (24h)

The goal of this course is to browse several aspects of asymptotic analysis in PDE. This will enable presenting a handful of PDE, among which Fokker-Planck equations, reaction-diffusion equations, Hamilton-Jacobi equations, kinetic equations, wave equations, and some integro-differential equations extending the previous ones. The course is also aimed to provide examples of PDE arising from physics and biology, together with some modeling background.

The following topics will be addressed during the course: - Long-time asymptotics of linear equations by entropy methods (either conservative or non-conservative) - Wave propagation in physics and biology, small wavelength asymptotics - Concentration dynamics in physics and biology, small variance asymptotics

The course will be illustrated by a collection of references (articles and books) in selected topics.

Evolutionary PDEs, Dragoş Iftimie (24h)

1. Some properties and reminders of distributions.
2. The Cauchy problem for linear PDEs.
   (a) Variable coefficients. Cauchy-Kovalevskaya theorem, characteristic hypersurfaces and Holmgren's uniqueness theorem. Well-posed problems.
   (b) Constant coefficients.
      - Existence of an elementary solution, the Malgrange-Ehrenpreis theorem. Examples.
      - Local resolubility of the Cauchy problem. Hyperbolicity. Gårding's theorem. Necessary and sufficient conditions for hyperbolicity.
3. Symmetric hyperbolic quasilinear systems.
4. Incompressible Euler equations. H^3 solutions and the Beale-Kato-Majda blow-up criterion.
5. Incompressible Navier-Stokes equations. Leray solutions. Uniqueness for small data in dimension 3.

Calculus of variations and elliptic partial differential equations and systems, Petru Mironescu (24h)

Description. This is an intermediate + course presenting some basic tools in the qualitative analysis, existence, and regularity theory for solutions of elliptic partial differential equations (PDEs). A first part, related to the direct method in the calculus of variations, goes beyond elliptic PDEs.

Prerequisites

1. Good knowledge of general measure theory and integration.
2. Reasonable knowledge of geometric aspects of the integration theory (Gauss-Ostrogradskii...) and of the local theory of submanifolds of $\mathbb{R}^n$.
3. Good knowledge of the basic results concerning the Laplace equation.

Syllabus

1. The direct method in the calculus of variations
   (a) Basic examples.
   (b) Notions of convexity.
   (c) Passing to the weak limits in nonlinear quantities. Compensation phenomena.
   (d) Gap phenomena.
2. Maximum principles and applications
   (a) Maximum principles for elliptic partial differential equations (PDEs) in non divergence and divergence form.
   (b) Iterative methods based on monotonicity (sub- and supersolutions).
   (c) Symmetry properties of solutions of semilinear elliptic PDEs.
   (d) Krasnoselskii's uniqueness result.
3. Regularity theory
   (a) Serrin's example.
   (b) Singular integrals.
   (c) L^p-theory for elliptic equations in non-divergence form.
   (d) A glimpse of the C^α-theory for elliptic equations in non-divergence form.
   (e) De Giorgi-Nash regularity theory for elliptic equations in divergence form.
   (f) Bootstrap. Regularity in the critical case.
   (g) A limiting case: Wente estimates. A glimpse of other compensation phenomena.
4. Other (non-)existence methods
   (a) Concentration-compactness.
   (b) Mountain pass solutions.
   (c) Other topological methods.
   (d) Pohozaev's identity.
5. A glimpse of phase-transition problems
   (a) A glimpse of the BV space.
   (b) Abstract Γ-convergence.
   (c) The Modica-Mortola functional in limit $\varepsilon \to 0$.
   (d) Vector-valued variants.

3 Advanced Courses

Optimal transport: introduction, applications and derivation, Aymeric Baradat (18h)

The starting point of this course will be to introduce the optimal transport problem between two probability measures on a physical space (such as subsets of Euclidean spaces) both in a static and in a dynamical version. We will see that the dynamical one gives a way to interpret formally the set of probability measures as an infinite dimensional Riemannian manifold, and we will discuss the corresponding notion of gradient flows as well as some properties of the geodesics. We will also see how the induced distance (the so-called Wasserstein distance) can be used to get contraction estimates in some nonlinear PDEs of Vlasov type.

In a last part, we will show how to recover optimal transport as the limit when the diffusivity tends to zero of the problem of minimizing the relative entropy w.r.t. the law of a Brownian motion under marginal constraints: this is the so-called entropic regularization of optimal transport. In addition to building a bridge between the theories of optimal transport and of large deviations of stochastic processes, this approach provides a way to compute efficiently approximated solutions thanks to the Sinkhorn algorithm.

The first two parts of the lecture will demand very few prerequisites except from notions of measure theory and basic knowledge in functional analysis. For the last part, it is better to be familiar with stochastic processes, and in particular with the Brownian motion.

The outline of the course will be as follows.

1. The static optimal transport problem
   - The Monge formulation and its relaxation, the Monge-Kantorovich problem
   - The Brenier theorem
   - The Wasserstein distance between probability measures
2. A dynamic reformulation: the Benamou-Brenier approach
   - The theory of the continuity equation
   - The Benamou-Brenier formulation of optimal transport
   - A formal introduction to gradient flows
   - Geodesic convexity and applications
   - The Wasserstein distance for nonlinear PDEs of Vlasov type
3. Entropy minimization w.r.t. the law of a Brownian motion
   - Statement of the Schrödinger problem
   - Static and Benamou-Brenier reformulation
   - The Sinkhorn algorithm
   - Γ-convergence towards optimal transport

I will provide a detailed bibliography during the lecture.

Compressible viscous flows with low or intermediate regularity, Didier Bresch (18h)

Motivation: The Navier-Stokes equations provide a basic mathematical model for describing the motion of a fluid. In the famous paper published in Acta Mathematica in 1934, "Sur le mouvement d'un fluide visqueux remplissant l'espace", Jean Leray (1906-1998) introduced the concept of weak solutions (and to do this, he also defined what is now called a Sobolev space), by giving a precise definition of what an irregular solution of the system is, and showed that there is such a weak solution for the homogeneous incompressible Navier-Stokes equations (see the Evolutinary PDEs basic course by D. Iftimie) .

We now call these solutions of minimal regularity (finite energy): solutions à la Leray. Even if the global existence of weak solutions does little about the well-posed character of the system, such an analysis has many practical interests and may help for intermediate regularity purposes. In addition to the physical meaning, because the regularity of the initial data assumed is minimal and strongly related to well-identified physical quantities, the stability properties of weak solutions on the continuous model help to better understand how to properly build stable numerical schemes that more often do not preserve strong regularity estimates (see works by T. Gallouet, R. Herbin, J.-C. Latché, E. Feireisl, T.G. Karper, A. Novotny and K. Saleh for instance). An other important kind of solutions (Hoff solutions) is the one with intermediate regularity where we allow density profile jumps with more informations on the velocity field.

Goal of the Course: This course aims at presenting some recent results related to the viscous compressible flows with low or intermediate regularity. We will compare the two notions of solutions. We will see that depending on the situations, different mathematical techniques have to be developed turning around transport equation with rough velocity fields.

Topics we plan to cover include:

- Compressible Navier-Stokes equations with density dependent viscosities: Non-linear hypocoercivity properties and energy-entropy weak solutions.
- Compressible Navier-Stokes equations with non-monotone pressure laws: Appropriate weights in a non-local compactness tool.
- Compressible Navier-Stokes equations with an anisotropic viscous tensor: Hoff solutions with intermediate regularity in an L^p framework.

We will explain the physical motivations before describing some mathematical tools on simplified systems. It could be nice to know the introductory book by L.C. Evans on compactness: Weak convergence methods for nonlinear partial differential equations. Volume 74, CBMS Regional Conference series in Math, 1990. I will provide a detailed bibliography during the lecture (see also the Refresher Course by M. De La Salle).

On the non linear Schrödinger equation, Nicolay Tzvetkov (18h)

We will consider the defocusing non linear Schrödinger equation. We will first show that in dimensions ≤ 3 this equations are globally well-posed in various geometric settings. As a by product , we will obtain that the H^1 norm of the solutions are bounded in time. We will then consider the question of the behaviour of the H^s, $s > 1$ norms of the solutions. This question is closely related to the possible migration of the Fourier modes of the solutions from low to high frequencies. We will show that when the problem is posed on the euclidean space $\mathbb{R}^3$ then the H^s, $s > 1$ norms remain bounded in time which prevents the possible migration to high frequencies of the Fourier modes of the solutions. In sharp contrast, we will show that when the problem is posed on the product space $\mathbb{T}^2 \times \mathbb{R}$ then the H^s, $s > 1$ norms of the solutions may be unbounded when the time evolves and thus the migration to higher modes may indeed occur. This phenomenon is some times referenced as a weak wave turbulence.

Plan of the course :

1. Dispersive estimates for the linear equation.
2. Global well-posedness in the energy space.
3. Large data scattering for NLS on $\mathbb{R}^3$.
4. The modified scattering.
5. The resonant system in the periodic setting and its large time analysis.
6. Solutions with unbounded Sobolev orbits for NLS on $\mathbb{T}^2 \times \mathbb{R}$.

Kinetic equations with collisions, Cédric Villani (18h)

A review of the kinetic theory with collision stemming from the works by Maxwell and Boltzmann, with a mixture of informal modeling, physical considerations, and analysis.

Program

1. The theory by Maxwell and Boltzmann
2. An overview of the problems and the variants
3. Lanford Theory
4. Analysis of the Boltzmann equation
5. $t \to \infty$
6. Collision theory for plasmas

References :

- C. Villani A Review of Mathematical Topics in Collisional Kinetic Theory, in *Handbook of Mathematical Fluid Dynamics*, Vol. 1, S. Friedlander, D. Serre (eds), pages 71–305, 2002.
- C. Villani Landau damping in *Modèles numériques pour la fusion*, Panorama et Synthèses, pages 237-326, 2013.