MASTER PROGRAM in MATHEMATICAL PHYSICS
at the University of Lyon for the academic year 2023/24

PROGRAM

Mise à jour (two weeks):
Preparation for the basic course 1 (10 hours) — J. Kellendonk
Preparation for the basic courses 2 and 3 (20 hours) — E. Legendre, K. Niederkrüger

Basic courses (3 x 24 hours):
Quantum Mechanics and Quantum Information Theory — G. Aubrun
Fiber Bundles in Differential Geometry and Gauge Theories — E. Legendre
Symplectic Geometry and Lie Groupoids — L. Ryvkin

Advanced courses (4 x 18 hours):
Topological Phases of Matter — J. Kellendonk
Quantum Field Theory and Renormalization — A. Frabetti
Seiberg-Witten Invariants — K. Niederkrüger
Poisson Sigma Models — T. Strobl*

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SUMMARY

Overall description of the program

Mathematical Physics, as a discipline within mathematics, has two main pillars: First, a development and/or application of mathematical tools so as to describe physical systems in a mathematically correct way. Second, it is the invention of mathematics as inspired by ideas coming from physics. There is one more important branch of Mathematical Physics where "toy models" of physical theories are studied: for example, reducing the dimension of spacetime often permits one to obtain a mathematical analogue of a physical theory that is accessible to a much more profound mathematical study—while keeping some of the characteristic features of the original system. The proposed program reflects all these aspects.

In one of three courses in the autumn semester, the axioms of *Quantum Mechanics* are introduced. Mathematically this requires tools from Hilbert space theory and Functional Analysis, some of the basics of which is reviewed/presented in a preparational course. Subsequently we turn to *Quantum Information Theory* , the physical relevance of which being underlined by the 2022 Nobel prize in physics.

The other two courses in the autumn semester deal with non-quantum, i.e. classical, systems: *Symplectic Geometry* and, more generally, Poisson geometry, is used in the Hamiltonian description of classical systems. *Riemannian geometry* and pseudo-Riemannian geometry is the basis of *General Relativity*. Essential for a mathematical description of fundamental interactions, moreover, are *Fiber bundles*: sections of particular vector bundles correspond to matter fields or particles, connections in principal bundles to interaction fields and forces between the particles. Since symmetries play an important role in many instances of physics, Lie groups, Lie group actions—and their joint generalization to *Lie groupoids*, of increasing importance in mathematics and physics—are introduced as well, together with their relation to Lie algebras and Lie algebroids. As a preparation to these two courses, there is a two-weeks' course on the basics of differential geometry.

In the spring semester the journey continues in the following way: *Topological Insulators* and, more generally, *Topological Phases of Matter* are among the exciting and fast developing new subjects in Mathematical Physics: Hand in hand with new experimental observations, in part sophisticated analysis is used for a quantum mechanical description of materials. In the corresponding course, after an introduction to the general ideas, a toy model of this in one spatial dimension is studied in detail.

Yang-Mills theories, introduced in one of the basic courses, are—except for electrodynamics—useful for describing nature only in their quantized version. This makes a course on *Quantum Field Theories* (QFTs) indispensable for an introduction to Mathematical Physics. Renormalization thechniques, invented by physicists to cure originally divergent integrals, were found to carry an interesting Hopf algebraic structure, as unraveled by A. Connes and D. Kreimer. The course introduces to the basics of perturbative QFT and renormalization techniques.

The remaining two courses, on *Seiberg-Witten Invariants* and *Poisson Sigma Models*, are good examples for mathematics that was invented by inspiration coming from methods used in physics: Studying particular Yang-Mills type gauge theories led Seiberg and Witten to discover ways of calculating topological and differential invariants for 4-manifolds, that became an interesting subject in *Differential Geometry* by itself. The perturbative quantization of the Poisson Sigma Model, a topological string theory using Poisson manifolds as target spaces, was used by Kontsevich to find a solution to a longstanding algebro-differential deformation problem. The Poisson sigma model also provides a toy model for coupled Gravity-Yang-Mills theories, namely when they are reduced to two spacetime dimensions. A study of the global causal structure found in this context then permits to introduce to some of the basic properties of black holes.

Short description of the seven main courses

Quantum Mechanics and Quantum Information Theory (by Guillaume Aubrun)

We present an axiomatic formulation of non-relativistic quantum mechanics and turn subsequently to quantum information theory.

Fiber Bundles in Differential Geometry and Gauge Theories (by Eveline Legendre)

We introduce the basics of (pseudo-)Riemannian geometry, of fiber bundles with connections, and of jet bundles. Their application to defining General Relativity and Yang-Mills-Higgs gauge theories is also part of the program.

Symplectic Geometry and Lie Groupoids (by Leonid Ryvkin)

In this course we investigate symplectic manifolds, as well as their generalizations to presymplectic and Poisson manifolds. We introduce the theory of Lie groupoids and Lie algebroids as generalized symmetries on smooth manifolds.

Topological Phases of Matter (by Johannes Kellendonk)

We consider topological phases of spin systems. These are defined by means of gapped short range entangled ground states. The aim of the course is to provide the necessary mathematical background to explain the classification of symmetry protected topological phases in one spatial dimension.

Quantum Field Theory and Renormalization (by Alessandra Frabetti)

This an introduction to perturbative Quantum Field Theory, the theory which describes interacting elementary particles like electrons with photons or self-interacting particles. We present standard tools such as Feynman graphs and path integrals, and we explain renormalization both in physical and mathematical terms.

Seiberg-Witten Invariants (by Klaus Niederkrüger)

The course is an introduction to a class of powerful invariants for smooth 4-manifolds. These invariants were obtained by the physicists Seiberg and Witten when studying the equations that describe the coupling of a certain spin particle with an electromagnetic field. Seiberg-Witten invariants simplified significantly the tools used previously in 4-dimensional topology.

Poisson Sigma Models (by Thomas Strobl)

We present the Poisson Sigma Model and study its basic features. We discuss two-dimensional gravity theories as a special case of the model, developing tools for understanding the global causal structure of the obtained spacetime manifolds. Finally, we discuss one of two possible applications: the Kontsevich formula for the deformation quantization of Poisson manifolds *or* the construction used in the integration of Lie algebroids to Lie groupoids.

DETAILED DESCRIPTION OF THE COURSES

1 Two weeks of preparations

1.1 Fundamentals of Hilbert spaces (10 hours, by Johannes Kellendonk)

- Hilbert space theory.
- Operators on Hilbert spaces, spectral theory.
- Elementary notions of operators algebras on Hilbert spaces (various topologies, C*-algebras and von Neumann algebras).
- Representations of symmetries on Hilbert spaces.

1.2 Calculus on manifolds (20 hours, by Eveline Legendre and Klaus Niederkrüger)

- Manifolds, tangent spaces, vector fields (flow and brackets). Frobenius Theorem. Tensor fields and their Lie derivatives.
- Differential forms, exterior derivative, Cartan calculus, de Rham cohomology.

2 Basic courses

2.1 Quantum Mechanics and Quantum Information Theory (24 hours by Guillaume Aubrun)

In the (shorter) first part of the course we first introduce the axioms of quantum mechanics: states, measurements, dynamics via Schrödinger's equation, and composite systems via tensor products. We describe the equivalence between open and closed quantum sytems via the concepts of a partial trace, completely positive operators, and we present the theorems of Choi, Kraus, Stinespring as well as of Naimark.

In the (longer) second part of the course, we focus on several topics relevant to the mathematical aspects of quantum information theory.

1. Quantum entanglement : entanglement criteria (positive partial transpose) and the quantum de Finetti theorem.
2. Nonlocality of quantum mechanics: Bell inequalities and their experimental violation, Grothendieck inequality.
3. Quantum information theory à la Shannon: quantum entropy, classical capacity of quantum channels, and quantum capacity of quantum channels.
4. Representation theory and quantum information: Schur-Weyl duality, spectrum estimates.

References:

- JM Landsberg, Quantum Computation and Quantum Information: a Mathematical Perspective (forthcoming book, should be published soon)
- Michael A. Nielsen and Isaac L. Chuang, Quantum Computation and Quantum Information
- G. Aubrun and S.J. Szarek, Alice and Bob meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory

2.2 Fiber Bundles in Differential Geometry and Gauge Theories (24 hours by Eveline Legendre)

Fiber bundles are natural objects to consider over manifolds. They play a central role in every parts of contemporary Differential Geometry and lie at the heart of a good part of Mathematical Physics. Therefore, a course introducing the basic and fundamental notions used to study fiber bundles (including connections, their curvature, caracteristic classes, ...) have become an essential ingredient for any modern graduate program in these fields. To introduce these notions is the main purpose of this course, where the presentation will be enhanced by motivations coming from Theoretical Physics.

Here is the plan of the lectures:

1. Fiber bundles and connections
2. Basics of Riemannian geometry
3. Jet bundles
4. Fundamentals about
   - General Relativity,
   - Electrodynamics,
   - Yang-Mills-Higgs gauge theories,
   - General gauge theories.

References:

- S.B Sontz, Principal Bundles (The Classical Case), Springer Universitext, (2015).
- S. Kobayashi; K. Nomizu. Foundations of Differential Geometry. Wiley Classics Library. Vol 1 (1963) and Vol 2 (1969)
- L. Fatibene, M. Francaviglia, Natural and Gauge Natural Formalism for Classical Field Theories, Springer, Dordrecht (2003)
- S. Gallot, D. Hulin, J. Lafontaine. Riemannian Geometry, Springer Universitext, (2004).

2.3 Symplectic Geometry and Lie Groupoids (24 hours by Leonid Ryvkin)

In the first part of the course, we introduce symplectic manifolds as the correct geometric setting to do classical mechanics from a Hamiltonian viewpoint. Investigating subsequently symmetries and conservation laws of physical theories, one discovers that not all symmetries can be described by the classical theory of Lie groups and Lie algebras.

The appropriate objects for this purpose, Lie groupoids and Lie algebroids, are of huge mathematical interest in themselves and investigating them constitutes the second part of the course. We discuss interesting examples, the structure theory, and the utility of these objects.

Finally, we turn to applications of Lie groupoids and Lie algebroids to symplectic geometry. In this context, we also take a look at Poisson structures, singular foliations, and Dirac structures.

References:

- Symplectic Geometry and Analytical Mechanics, by Libermann and Marle
- Poisson Structures by Camille Laurent-Gengoux, Anne Pichereau, Pol Vanhaecke
- Lie Groupoids and Lie algebroids, lecture notes by Eckhard Meinrenken, (https://www.math.toronto.edu/mein/teaching/MAT1341_LieGroupoids/Groupoids.pdf)

3 Advanced courses

3.1 Topological phases of matter (18 hours by Johannes Kellendonk)

The notion of a topological phase in physics is relatively new. The general principle is that distinct physical systems are considered to be in the same topological phase if they are in some sense homotopic. There are different contexts and approaches in which such a notion can be made precise. In this course, we consider topological phases of spin systems. These are defined by means of gapped short range entangled ground states. The aim of the course is to provide the necessary mathematical background to explain the classification of symmetry protected topological phases in one dimension. The corresponding question in higher dimensions is a very active area of current research.

The mathematical background taught in this course includes the basic notions and properties of C*- and von Neumann algebras and how symmetries are represented on them. Here we follow some of the material of the two books by Bratteli and Robinson “Operator algebras and quantum statistical mechanics; vol. 1 and 2”. This is complemented by the current research literature. Below two such articles, with overlapping content that may serve as a source for the relevant material for the course. Besides operator algebra techniques, group cohomology also plays a central role.

References:

- Bratteli, Ola, and Derek William Robinson. Operator algebras and quantum statistical mechanics: Volumes 1 and 2. Springer.
- Kapustin, Anton, Nikita Sopenko, and Bowen Yang. "A classification of invertible phases of bosonic quantum lattice systems in one dimension." Journal of Mathematical Physics 62, no. 8 (2021): 081901.
- Ogata, Yoshiko. "A classification of pure states on quantum spin chains satisfying the split property with on-site finite group symmetries." Transactions of the American Mathematical Society, Series B 8, no. 2 (2021): 39-65.

3.2 Quantum Field Theory and Renormalization (18 hours by Alessandra Frabetti)

The aim of the lectures is twofold. First we present the essential tools of (perturbative) quantum field theory: free propagators, Feynman graphs, correlation functions for interacting theories, Dyson-Schwinger equations, divergencies, counterterms and renormalization formulas. Then we present some algebraic and combinatorical tools which simplify the presentation of renormalization formulas (namely proalgebraic groups and Hopf algebras).

The lectures include a brief introduction to field theory, examples and exercices on the Klein-Gordon, Dirac, Maxwell and Yang-Mills theories as well as on some basic algebraic and proalgebraic groups with relative Hopf algebras.

The topics presented in these lectures can be further developped in applications to physics (in renormalization theory) and in algebraic topics (proalgebraic loops, combinatorial Hopf algebras, operads).

References:

- E. Abe. Hopf Algebras. Cambridge University Press 1980.
- J.C. Collins. Renormalization. Cambridge University Press 1984.
- A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem. Commun. Math. Phys., 210 (2000) 249–273.
- A. Connes and D. Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem II: the beta function, diffeomorphisms and the renormalization group. Commun. Math. Phys., 216 (2001) 215-241.
- W.N. Cottingham end D.A. Greenwood. An Introduction to the Standard Model of Particle Physics. Cambridge University Press 1998.
- A. Frabetti. Renormalization Hopf algebras and combinatorial groups. In "Geometric and Topological Methods for Quantum Field Theory", Cambridge University Press 2010.
- A. Frabetti and D. Manchon. Five interpretations of Faà di Bruno's formula. In "Dyson-Schwinger Equations and Faà di Bruno Hopf Algebras in Physics and Combinatorics", IRMA Lectures in Mathematics and Theoretical Physics Vol. 21, European Mathematical Society, 2015.
- C. Itzykson and J.-B. Zuber. Quantum Field Theory. McGaw-Hill Ed., 1980.
- M.E. Peskin and D.V. Schroeder. An Introduction to Quantum Field Theory. Perseus Books Pub. L.L.C. 1995.
- M.E. Sweedler. Hopf algebras. Mathematics Lecture Note Series, W. A. Benjamin Inc. 1969.

3.3 Seiberg-Witten Invariants (18 hours by Klaus Niederkrüger)

Most classical topological invariants like homology and the fundamental group do not detect much more than the homotopy type of the considered space. Donaldson, Floer, Gromov, Seiberg, Witten, and many others developed, in the eighties and nineties of the last century, a new class of much finer invariants that can, for example, distinguish different smooth structures on a given manifold.

While each of these theories is very different, and easily allows a large part of a researcher's life to be spent on its details, the underlying idea is always similar: choose some type of "natural" partial differential equation and study its solutions. Setting up such a PDE requires several auxiliary choices to be made (for example the choice of a Riemannian metric). An important step of this strategy requires then to extract the information that is independent of any of the auxiliary choices from the solution space of the PDE.

Obviously, a random PDE does not yield any meaningful theory. In practice, even if we are only interested in the mathematical aspects of geometry, the most successful strategy has been to use equations from physics. In this course, we want to give an introduction to Seiberg-Witten invariants, which are used in the study of smooth 4-manifolds.

Named after its creators, two physicists, this theory yields most of the results of Donaldson theory, but takes away much of the technical complications of the latter one. The considered equations are those of a massless spin particle that is coupled to an electromagnetic field: The field is specified by a connection of a principal bundle, the particle is a section in a spinor bundle. In the first part of the course, we introduce the necessary mathematical notions needed to setup the Seiberg-Witten equations, like for example spinor bundles and Dirac operators. Here some of the underlying physical intution is commented upon.

In the second part of the course, we review the analytical basics needed to describe the properties of the solution space.

In the last part, we study examples where we are able to determine the invariants: Finding the solutions of a given PDE explicitly is of course in general not possible. It is thus interesting to see how one can determine, in certain cases, such solutions and how to understand the type of information that we can extract from them.

The underlying motivation for this course is to show how modern theoretical physics interacts with mathematics. In the specific case of Seiberg-Witten invariants, it is hard to imagine that anybody without the necessary physical intuition would have been able to propose such a theory.

References:

- E. Witten, "Monopoles and four-manifolds.", Mathematical Research Letters, 1 (6): 769-796.
- J.D. Moore, "Lectures on Seiberg-Witten invariants", Lecture Notes in Mathematics, vol. 1629, Springer.
- J. Morgan, "The Seiberg-Witten equations and applications to the topology of smooth four-manifolds", Mathematical Notes, vol. 44, Princeton University Press.

3.4 Poisson Sigma Models (18 hours by Thomas Strobl)

We introduce Yang-Mills and gravitational theories like R^2-gravity on 2-dimensional spacetimes. We show that such models of physical theories—and in fact a much larger class of them called generalized dilaton gravities coupled to Yang-Mills theories—can be reformulated as particular Poisson sigma models (PSM)s.

In general, a PSM is given in terms of a canonical functional S_{PSM} defined on the vector bundle morphisms ψ from TN to T^*M, where N is an orientable 2-manifold and M a Poisson manifold. We study the first basic properties of these models: we show that the Euler-Lagrange equations of S_{PSM} restrict ψ to be a morphism of Lie algebroids and that gauge transformations of such solutions are given by Lie algebroid homotopies. In this context, $\mathbb{Z}$-graded geometry is introduced, since it simplifies the definition of such morphisms and homotopies significantly.

Using the Weinstein splitting theorem, proven in the basic course by Ryvkin, we construct all local solutions of the initially mentioned coupled gravity-Yang-Mills theories, which can be obtained in closed form in two dimensions. We subsequently construct all maximally extended, universal coverings of such spacetime solutions, using the technique of Penrose diagrams—the global causal structures encountered contain various black hole scenarios, as one finds. Time permitting, we also provide an explicit presentation of the isometry group G for each of such a spacetime solution. Studying appropriate subgroups of G, one arrives at a description of the moduli space of classical solutions up to diffeomorphisms for arbitrary topologies of the 2-manifold N.

In a final part of this course, we provide insight into one of the following two subjects, the choice depending on the wishes of participants. Each of these is a construction, suggested by the PSM, that led to a solution of a longstanding and important problem in mathematics:

1. The integration problem of Lie algebroids to Lie groupoids, solved by Crainic and Fernandes.
2. An explicit construction, given by Kontsevich, of all formal associative deformations of the product of functions on a manifold or, equivariantly, the deformation quantization of arbitrary Poisson manifolds.

For 1, the construction, discovered by Cattaneo and Felder, results from a coisotropic reduction of the Hamiltonian formulation of the PSM. Albeit defined in infinite dimensions, the reduction, if smooth, gives a finite-dimensional symplectic manifold which carries the structure of a Lie groupoid. This Lie groupoid is the one integrating the Lie algebroid T^*M associated to the Poisson manifold M in the sigma model. Crainic and Fernandes used a straightforward generalization of this construction to arbitrary Lie algebroids, which simultaneously generalizes the integration of Lie algebras to Lie groups using homotopy classes of paths, encountered already in the basic course by Ryvkin. Identifying the necessary and sufficient conditions for smoothness, gives their celebrated integration result.

For 2, one studies the perturbative quantization of the PSM. Due to the gauge invariance, this requires the use of (infinite dimensional) graded geometry within the BV or AKSZ formalism, which we briefly outline. The Feynman diagrams of the PSM then become what are now called the Kontsevich graphs. The deformed, non-commutative product of functions on M is obtained from correlation functions of the quantum PSM. That it is associative is suggested by the topological nature of the PSM, but then proven by Kontsevich in purely mathematical terms. As for Seiberg-Witten theory, also the construction of Kontsevich is unimaginable without the inspiration and the tools coming from physics.

References:

- A. Cattaneo and G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000) 591-611.
- A. Cattaneo and G. Felder, Poisson sigma models and symplectic groupoids, Progress in Mathematics 198, 61-93 (Birkhäuser, 2001).
- M. Crainic and R. Fernandes, Integrability of Lie brackets, Ann. of Math., Vol. 157 (2003), no. 2, 575–620.
- T. Klösch and T. Strobl, Classical and Quantum Gravity in $1+1$ Dimensions, Part II: The Universal Coverings, Class. Quantum Grav. 13 (1996) 2395-2421.
- T. Klösch and T. Strobl, Classical and Quantum Gravity in $1+1$ Dimensions, Part III: Solutions of Arbitrary Topology in 1+1 Gravity, Class. Quantum Grav. 14 (1997), 1689-1723.
- M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys., 66 (2003), 157-216.