Proposal for a MA2 program ICJ - UMPA
École Doctorale 512
Groups, Geometry, Dynamics and Model Theory

November 21, 2020

The program consists of eight courses and cover a rather broad range of mathematics between group theory, dynamical systems, geometry and model theory. There are four courses at the fall semester and four at the spring semester. A student following the program is expected to choose three courses out of four in each semester.

Basic courses

$*$ **Introduction to ergodic theory and topological dynamics (Damien Gaboriau and Adrien Le Boudec), 24h**. This course will be an introduction to the ergodic theory and topological dynamics of infinite discrete groups. The following topics will be addressed:

- measure preserving actions, existence of invariant measure for actions of the group Z, ergodic theorems, ergodicity, ergodic decomposition;
- recurrence, minimality, existence of minimal sub-systems;
- notions of mixing;
- introduction to unitary representations, connection with the previous notions;
- entropy;
- symbolic dynamics;
- Rokhlin lemma; full groups ; Dye theorem and Ornstein-Weiss theorem.

$*$ **Lie groups and Lie algebras (Sophie Morel and Bruno Sévennec) 24h**.

- Definition of topological groups, of Lie groups and their Lie algebras, morphisms of Lie groups and their differentials.
- Example of closed subgroups of $\mathrm{GL}_n(\mathbf{C})$.
- Universal covers and fundamental groups of topological groups and Lie groups.
- Lie subgroups, the closed subgroup theorem, Lie subalgebras vs Lie subgroups.
- Morphisms of Lie algebras vs morphisms of Lie groups.

- Representations of topological and Lie groups, unitary representations, representations of Lie algebras.
- Haar measure, representations of compact groups are unitarizable, hence semisimple.
- Example of finite-dimensional representations of $\mathfrak{sl}_2(\mathbf{C})$ and $\mathrm{SU}(2)$. Application to finite-dimensional representations of $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{C})$.
- Finite-dimensional representations of compact groups: characters, Schur orthogonality, Peter-Weyl theorem. Application: every compact Lie group is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbf{C})$.
- The semisimple Lie algebra $\mathfrak{sl}_n(\mathbf{C})$ and the corresponding Lie groups $\mathrm{SU}(n)$, $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{SL}_n(\mathbf{C})$: semisimplicity of finite-dimensional representations, Iwasawa decomposition, Jordan-Chevalley decomposition, Jacobson-Morozov theorem, roots, classification of representations by highest weights, time permitting Weyl character formula.
- (Time permitting) General complex semisimple Lie algebras: Existence of compact real forms and of a corresponding compact Lie group. Applications.

Sample reading list: Mark Sepanski, *Compact Lie groups*

$*$ **Geometric group theory (Jean-Claude Sikorav), 24h**. Geometric group theory can be said to have been founded around 1880 at the same time as abstract group theory itself, with the works of F. Klein on Fuchsian groups and the construction of the free group by W. Dyck as a Schottky group of hyperbolic transformations.

For a long time, under the name "Combinatorial group theory", the subject was more connected to topology than to geometry, although the works of M. Dehn in the 1900s which led him to formulate his famous decision problems for finitely presented groups had a strong geometrical flavor.

The crowning achievements of this combinatorial-topological approach were the algorithmic unsolvability of the word problem for finitely presented groups by P.S. Novikov in 1955, and the small cancellation theory developed in the 1950s and and 1960s by V.A. Tartakovsky, M. Greendlinger and others. This approach has continued to flourish up to now, for instance with the Bass-Serre theory of groups acting on trees and the works of A. Yu. Olshanskii and his school.

But geometrical ideas started to come back in the 1950s, with the work of A. Schwarz on the growth of finitely generated groups and the isoperimetrical interpretation of amenability by Følner. And with the works of Stallings on the theory of ends and on finiteness properties. One should also note the “isoperimetric” reformulation of the word problem in terms of the Dehn function.

The main impulsion for geometrical methods in group theory came from Gromov in the 1970s and 1980s, with his celebrated results on polynomial groups and nilpotent groups, and the theory of hyperbolic groups. From then onwards they have flourished wonderfully, with ever increasing connections to other fields, including three-dimensional topology, logic, computer science and probabilitiy.

The aim of this course is to introduce some of the main concepts alluded to above:

- finitely presentations of groups, Cayley graphs, word problem, Dehn function

- free products, amalgamated free products, HNN extensions
- Bass-Serre theory
- theory of ends, finiteness properties
- growth function of finitely generated groups, amenability
- hyperbolic groups, CAT(0) groups.

References

[1] G. Baumslag, *Topics in Combinatorial Group Theory*, Birkhäuser Lecture in Mathematics, 1993.

[2] M. Coornaert, T. Delzant et A. Papadopoulous, *Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov*, Springer Lecture Notes in Mathematics 1441, 1990.

[3] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Grouop Theory 1, Niblo and Roiter ed., London Math. Soc. Lecture Notes 182, 1-295.

[4] P. de la Harpe, *Topics in Geometric Group theory*, Chicago Lectures in Math., The University of Chicago Press, 2000.

[5] R.C. Lyndon and P.E. Schupp *Combinatorial Group Theory*, Springer Classics in Mathematics, 2001.

[6] J.-P. Serre, *Arbres, amalgames,* SL(2), Astérisque 46, 1977. English translation: *Trees*, Springer, 1980.

$*$ **Model Theory and its applications (Itaï Ben Yaacov), 24h**. The goal of the course is to provide an introduction to modern Model Theory, with some attention given to its interactions with other domains present in the M2, especially in the Geometry and Dynamics programme.

Foundations

- Dualities of Stone and Gelfand.
- Ultraproducts.
- Formulas and space types. The Compactness Theorem.
- Quantifier elimination.
- Type omission via Baire's Theorem.
- Countable / separable categoricity, automorphism groups.
- Roelcke precompact groups.

Optional themes

We will certainly not treat everything listed here. Some of the items may be the subject of home assignments.

- Special theories: existential, universal, inductive, model-complete.
- Examples of applications to algebra: Nullstellensatz via quantifier elimination, uniformity of the membership problem for a radical polynomial ideal.
- Fraïssé classes: discrete and continuous.
- Graphons – Szemeredi's Lemma and applications
- Ramsey Theory and extremely amenable groups.

Advanced courses

∗ **Amenability and dynamics (Nicolás Matte Bon and Todor Tsankov), 24h**. The course will begin by introducing the notion of amenability of groups, and then present several aspects in the dynamics of group actions on topological and measured spaces in which this notion plays a central role. This will be in part as pretext to introduce some current domains of research on infinite groups and dynamics. Topics that will be covered include:

- Amenability and some of its equivalent characterisations. Application to the Banach–Tarski paradox.
- The Furstenberg boundary.
- Connections of amenability to random walks on groups and the Poisson boundary.
- Amenability of topological full groups of minimal homeomorphisms of the Cantor set (Juschenko–Monod Theorem).
- Følner tilings (Quasi-tiling lemma and Downarowicz-Huczeck-Zhang finite tileability theorem).
- Amenability of measured equivalence relation. Applications to non-amenability of groups of piecewise projective homeomorphisms. Orbit-equivalence of actions of amenable groups (Ornstein–Weiss theorem)

∗ **Lattices in semisimple Lie groups (Amine Marrakchi and Mikael de la Salle), 24h**. The course will give an introduction to the study of lattices in semisimple Lie groups and their representation theory, with applications to expander graphs and ergodic theory. The following topics will be presented :

- The Borel-Harish-Chandra theorem on arithmetic subgroups.
- Kazhdan's property (T)
- Expander graphs
- The Howe-Moore property

- Actions on Banach spaces, Bader-Furman-Gelander-Monod theorem

Depending on the time and the audience's taste, we may also developp further the ergodic theory of arithmetic lattices (Stuck-Zimmer theorem) and/or further representation theory (Lafforgue's strong property (T)).

$*$ **Actions on trees and the elementary theory of free groups (Abderezak Ould Houcine), 24h** The subject of the course lies at the intersection of geometric group theory and model theory of groups. The main theme of geometric group theory is the study of groups as geometric objects (metric spaces or acting on metric spaces). In this area, group actions on trees is one of the most active field since several decades. Bass-Serre theory, which quickly became standard in geometric group theory, shows that a group acts nontrivially by automorphisms on a simplicial tree if and only if it is an iterated decomposition as amalgamated free products and HNN-extensions. Bass-Serre theory gives a complete description of groups acting nontrivially by automorphisms on simplicial trees.

After Bass-Serre theory, it becomes natural to consider the wider context of group actions by isometries on real trees, where actions by automorphisms on simplicial trees can be seen as a particular case. The subject was then more developed when Rips proved Morgan-Shalen's conjecture on the structure of groups acting freely on real trees.

Regarding model theory, it has as central theme the study of "algebraic objects" in a very general framework. It is a branch of mathematical logic which studies, among others, relations between a formal language and models or mathematical structures. For example, basic mathematical structures of algebra such as groups, rings, fields and vector spaces fall within this framework. Model theory of groups is a part of model theory which is devoted to study model-theoretic properties of groups. The elementary theory of a given group G is the set of all sentences ϕ such that G satisfies ϕ. Two groups are said to be elementary equivalent if they have the same elementary theory.

The most celebrated problem in this field is the famous Tarski's conjecture, formulated by Tarski around 1945 and which states that any tow nonabelian free groups are elementary equivalent. A proof of the conjecture was given by Kharlampovich-Myasnikov and independently by Sela around 1998-2006. Sela uses tools from geometric group theory, more particularly actions on trees. In that context one needs to understand solutions of equations over free groups. A principal ingredient in Sela's approach is the use of actions on simplicial trees and their limits in the Gromov-Hausdorff topology, decompositions of groups acting on real trees, JSJ-decompositions and the shortening argument.

The aim of this course is to provide a study of groups acting on real trees, an already intrinsically very rich framework, and an introduction to the basic building blocks of the elementary theory of free groups.

Content

- Bass-Serre theory (A review)
- Actions on real trees
- Gromov-Hausdorff limits & Asymptotic cones
- JSJ-decompositions
- The shortening argument
- Limit groups and their actions

- Constructibility of limit groups & diagrams
- The $\forall\exists$-theory and the Homogeneity.

References

[1] A. Ould Houcine, *Homogeneity and prime models in torsion-free hyperbolic groups.* Confluentes Mathematici, 3 (1) (2011) 121-155.

[2] Z. Sela, *Diophantine geometry over groups I: Makanin-Razborov diagrams.* Publications Mathematiques de l'IHES 93 (2001), 31-105.

[3] J.-P. Serre, *Arbres, amalgames,* SL(2), Astérisque 46, 1977. English translation: *Trees*, Springer, 1980.

∗ **Model Theory of Groups (Frank O. Wagner) 24h.** This course will introduce various aspects of the model theory of groups, possibly in an enriched langage (including in particular modules, rings and fields). We shall in particular study various tameness conditions commonly studied in model theory, and their consequences in the group context. In the context of pseudfinite groups, we shall give the model-theoretic background leading to the Breuillard-Green-Tao classification of approximate groups.

Programme:

- Abelian groups and modules, pp-elimination of quantifiers and weak normality.
- Pseudofinite simple groups
- Approximate groups
- Minimal groups and fields
- Tameness conditions : stability, dependence (NIP), NTP2 and associated ranks
- Genericity and generic types
- Stabilizers
- Chain conditions
- ω-categorical groups