GROUPS AND DYNAMICS

MA2 2023-2024

This program offers courses on both classical and modern topics around group theory and dynamics, some of them also having important geometric features.

Refresher courses will ensure a common knowledge for students with various mathematical backgrounds. The first semester consists of three fundamental courses. The second semester consists of four advanced courses, and students have to choose three of them.

Contents

Refresher courses

0.1. **Riemann surfaces.** (6h) Aurélien Alvarez

0.2. **Topics in the theory of infinite groups.** (6h) Adrien Le Boudec

0.3. **Topics in Riemannian geometry.** (12h) Jean-Claude Sikorav - Ghani Zeghib

Semester 1

1.1. **Representation theory.** (24h) (Sophie Morel - Bruno Sévennec)
TBA

1.2. **Geometric group theory.** (24h) (Adrien Le Boudec)

One aspect of geometric group theory is to study infinite groups by viewing them as geometric objects. This course will introduce some important concepts and fundamental results in this area. Topics that will be covered include:

- Invariant metrics on groups, word metrics, graphs associated to groups, ends of groups.
- Large scale geometry: group actions by isometries, isometry groups of proper metric spaces, Milnor-Schwarz lemma, quasi-isometries, quasi-isometry invariants.

- Group actions on trees: individual isometries, types of actions, ping-pong, amalgamated products and HNN-extensions, ends and group splittings, groups quasi-isometric to trees, property (FA).
- Growth of groups: Gromov theorem on discrete groups of polynomial growth, locally compact groups of polynomial growth, groups of exponential growth.
- Time permitting: Tits alternative, amenability, groups of non-positive curvature, asymptotic dimension,...

References

- C. Druţu and M. Kapovich, *Geometric group theory*, American Mathematical Society, 2018.
- Y. Cornulier and P. de la Harpe, *Metric geometry of locally compact groups*, European Mathematical Society (EMS), 2016.
- M. Gromov, *Asymptotic invariants of infinite groups*, Cambridge Univ. Press, Cambridge, 1993.

1.3. **Introduction to dynamical systems and ergodic theory.** (24h) (Jean-Claude Sikorav)

The main part of the course will be focused on topological dynamics, by which we mean mostly homeomorphisms of compact spaces, diffeomorphisms of compact manifolds and flows on compact manifolds. It will involve the following topics:

- General concepts, including various notions of recurrence
- Low-dimensional phenomena: homeomorphisms of the Cantor set, in particular symbolic dynamics, homeomorphisms and diffeomorphisms of the circle
- Hyperbolicity, from periodic points to Anosov diffeomorphisms and flows, relation with symbolic dynamics
- Genericity and structural stability, notably Morse-Smale and Anosov systems.

There will also be an introduction to ergodic theory, mostly the study of transformations preserving a probability measure (not necessarily invertible), and present the following topics:

- Ergodicity, Birkhoff ergodic theorem
- Measure-theoretic entropy
- Topological entropy, variational principle.

References

- E. Akin, *The General Topology of Dynamical Systems*, Amer. Math. Soc. Grad. Studies in Math., 1993.
- M. Brin and G. Stuck, *Introduction to dynamical Systems.* Corrected paper back edition of the 2002 original. Cambridge Univ. press, 2015.
- A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Smooth Dynamical Systems*, Cambridge University Press, 1995.

Semester 2

2.1. **Amenability and orbit equivalence.** (24h) (Damien Gaboriau - Todor Tsankov)

The first part of the course will give an introduction to the theory of amenability and dynamics of amenable groups. It will continue with an introduction to orbit equivalence theory of probability measure-preserving actions and contrast the behaviour of amenable and non-amenable groups in this setting. This will be an opportunity to discuss aspects of measured group theory. Some rigidity results will be discussed, and, time permitting, applications to probability theory of percolation.

The following topics will be covered:

- Equivalent definitions of amenability
- The Banach-Tarski paradox
- Group actions on probability spaces and orbit equivalence
- Hyperfiniteness and the Connes-Feldman-Weiss theorem
- Graphings and cost
- Non-orbit equivalence results for free groups
- Property (T) and orbit equivalence rigidity
- Applications to percolation

2.2. **Groups arising from dynamical systems.** (24h) (Nicolás MATTE BON)

The course will present two constructions of groups associated to dynamical systems: iterated monodromy groups of complex rational maps and topological full groups of symbolic dynamical systems.

The notion of iterated monodromy group allows to encode every expanding self-map of a topological space into a group acing on a rooted tree, which remembers completely the dynamics of iterations of the map. An important special example is the case of iterations of a rational function on one complex variable: the iterated monodromy group allows to describe the dynamics of the map on its Julia set. Topological full groups are groups associated to every homeomorphism (or group of homeomorphisms) of the Cantor set, which encode the local behaviour of the homeomorphism.

In both cases the properties of the resulting group depend in a rich way on the properties of the underlying dynamical system. These constructions share the feature to groups with very interesting properties in relation to important concepts in modern group theory such as amenability, growth, and the behaviour of random walk. The course will be in part a pretext to study these topics.

Prerequisites: having attended an introductory course to dynamical systems. Some basic probability theory (e.g. the notion of Markov chain) might be useful but should not be essential.

References

- V. Nekrashevych, *Groups and Topological Dynamics*, volume 223 of Graduate Studies in Mathematics. Amer. Math. Soc., Providence, RI, 2022.

2.3. **Dynamics of complex differential equations.** (24h) (Aurélien ALVAREZ - Ghani ZEGHIB)

The study of differential equations in the complex domain is a relatively old subject which dates back to the end of the 19th century but continues to arouse great interest. These are differentiable equations with a complex variable and algebraic coefficients. After compactification, we come back to the study of algebraic (singular) foliations on $\mathrm{P}^2_{\mathbf{C}}$. The aim of the course is to study the dynamics and the geometry of certain classes of such foliations. The main ingredient is monodromy which is a linear representation of the fundamental group of the complement of the set of singularities. Some of the topics that will be covered in this course include

- linear case: Fuchsian equations, Fuchsian groups, RIEMANN-HILBERT problem (monodromy inverse problem);
- the differential equations of RICCATI and their monodromies;
- the equations of PAINLEVÉ and their applications in various mathematical fields;
- the algebraic structure of the foliation space of the projective plane $\mathrm{P}^2_{\mathbf{C}}$ and (time permitting) the structural stability of the equation of JOUANOLOU.

References

- N. Bergeron, *Differential equations with algebraic coefficients over arithmetic manifolds.* The scientific legacy of Poincaré, 47–71, Hist. Math., 36, Amer. Math. Soc., Providence, RI, 2010.
- F. Beukers, Notes on differential equations and hypergeometric functions.
- F. Cano, D. Cerveau, J. Déserti, *Théorie élémentaire des feuilletages holomorphes singuliers*, Belin, 2013.
- H. P. de Saint-Gervais, *Uniformisation des surfaces de Riemann*, ENS Éditions, 2010.
- R. Tazzioli Fonctions fuchsiennes ou schwarziennes? Mieux poincaréennes, *Images des mathématiques*, 2010.

2.4. **Geodesic flows.** (24h) (Marco Mazzucchelli)

Geodesic flows are one of the most important class of conservative dynamical systems. Poincaré began their study as a toy model for the more complicated dynamical systems arising in celestial mechanics. As it turned out, geodesic flows posed many challenges, some of which still unresolved, motivated the birth of critical point theory, and ultimately inspired certain developments in symplectic topology.

The course will began with an introduction to geodesic flows, which will require some basic symplectic geometry of tangent bundles. Next, the course will present the proof of a few celebrated theorems, ranging from Birkhoff's first results (~ 1915) to very recent ones (~ 2020), time permitting.

The students will need some basic notions of Riemannian geometry (Riemannian metrics, definition of geodesic), dynamical systems (Jean-Claude Sikorav's course will be largely enough), and ideally some algebraic topology (basic properties of homology and cohomology).

A tentative program will be the following:

- Introduction to geodesic flows.
- Topological entropy of geodesic flows: Mañé's formula
- Existence of closed geodesics: Birkhoff theorem.
- Curve shortening flow, and a proof of the three simple closed geodesics theorem of Lusternik-Schnirelmann.
- Multiplicity of closed geodesics: theorems of Gromoll-Meyer and Bangert-Franks-Hingston.
- Existence of surfaces of sections for geodesic flows, and some dynamical applications: proof of the structural stability conjecture.
- Brief introduction to geometric inverse problems involving geodesics: spectral rigidity, boundary rigidity, and lens rigidity.

References

- G. P. Paternain, *Geodesic flows*, Progress in Mathematics, vol. 180, Birkhäuser Boston, Inc., Boston, MA, 1999.
- C. Guillarmou and M. Mazzucchelli, *An introduction to Geometric Inverse Problems*, forthcoming book.
- A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Smooth Dynamical Systems*, Cambridge University Press, 1995.
- W. Klingenberg, *Riemannian geometry*, Walter de Gruyter, 1995.
- W. Klingenberg, *Lectures on closed geodesics*, Springer, 1978.