Topics in Complex Algebraic, Kähler and Symplectic geometries.

Master 2 program of geometry in Lyon

November 1, 2024

This program offers courses on both classical and modern topics around Kähler geometry, complex algebraic geometry and symplectic topology. Refresher courses will ensure a common knowledge for students with various mathematical backgrounds. The first semester consists of four fundamental courses and the second semester consists of three advanced courses.

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0 Refresher courses

0.1 Several complex variables (10h)

Jean-Claude Sikorav

0.2 Complex algebraic geometry (10h)

Nicolas Ressayre

0.3 Differential topology (10h)

Klaus Niederkrüger

1 Basic courses : semester 1

1.1 Introduction to Kähler geometry (24h)

Jean-Claude Sikorav

The course will introduce some of the main ideas of Complex Geometry, which is the geometric version of Functions of Several Complex Variables, and more specifically of Kähler geometry, which is an essential tool in many questions of Complex Algebraic Geometry.

We hope to cover the following topics:

- Differential calculus on complex manifolds, Dolbeault cohomology
- Analytic sets, application: Chow's theorem
- Complex and holomorphic lines bundles, Chern class
- Connection on a line bundle, curvature, positivity
- Kähler metrics, Kähler identities
- Hodge theory and cohomology of compact Kähler manifolds
- Curvature tensor and Ricci form of a Kähler manifold

The main prerequisites are the basics of functions of several complex variables, for which a refresher course will be offered.

References

[1] J.-P. Demailly, *Complex Analytic and Differential Geometry*, https: www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf.

[2] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.

1.2 Introduction to Complex Algebraic Geometry (24h)

Antoine Etesse

The goal of this course is to introduce several important notions in (complex) algebraic geometry.

The first important *invariants*[1] that one associates to an algebraic variety is the cohomology groups of its structural sheaf. We will therefore start off with the general notions of *sheaves* and *cohomology of sheaves*.

Beside the structural sheaf on an algebraic variety X (which should be thought of as the trivial line bundle on X), one is interested in the set of (isomorphism classes of) line bundles on X, called the *Picard group*. We will therefore introduce this set, and see how this is connected to codimension 1 subvarieties of X, leading to the notions of *Weil divisors* and *Cartier divisors*.

Almost by definition, an algebraic variety X admits a line bundle $L \to X$ that is called *ample*. We will discuss this important notion, and its interpretations. In particular, we will see its analytic interpretation, culminating to the famous *Kodaïra embedding theorem*.

References

[1] R. Lazarseld, *Positivity in Algebraic Geometry I*, Springer, 2004.

[2] C. Voisin, *Hodge Theory and Complex Algebraic Geometry*, Cambridge University Press, 2002.

1.3 Convexity in symplectic geometry (12h)

Klaus Niederkrüger

The aim of this course is to give an introduction to Lie group actions on symplectic manifolds. We'll briefly cover the formalism of Hamiltonian functions and moment maps. Concentrating on compact Lie groups, we prove the symplectic slice theorem which provides the local

[1] In a vague sense!

model for orbits. We will show that the Hamiltonian function of a circle action is of Morse-Bott type, and that all fixed point components are symplectic manifolds. The ultimate goal is to combine this information to prove the Atiyah-Guillemin-Sternberg theorem that states that the image of the moment map of a Hamiltonian torus action is a convex polytope.

References

[1] M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. 14 (1982), 1-15, Springer, 2004.

[2] V. Guillemin, S. Sternberg, *Convexity properties of the moment mapping*, Invent Math 67, 491–513 (1982)

[3] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs.

1.4 Reductive algebraic group over $\mathbb{C}$ (12h)

Jérôme Germoni

In this mini-course, we study the complex reductive groups, their structure and their representation theory. A complex reductive group is defined to be the complexified of a real compact Lie reductive. We will show that their structure is very rigid and his governed by a combinatorial data called root systems. Their representations are semi-simple and the irreducible ones are bijectively parametrized integer points in a convex cone. By lack of time, some proofs will made only for the linear group $\mathrm{GL}_n(\mathbb{C})$.

[1] J. Humphreys, *Linear algebraic Groups*, Graduate text in Mathematics, Springer 1975.

[2] Fulton-Harris, *Representation Theory. A first Course*, Graduate text in Mathematics, Springer 2004.

2 Advanced courses : semester 2

2.1 GIT and Kempf-Ness theorem (24h)

Nicolas Ressayre

In this course, we study the basic Geometric Invariant Theory (GIT). We work in a context of complex algebraic geometry: a complex reductive group (such as $\mathrm{GL}_n(\mathbb{C})$) acts on a projective variety (such as the projective space $\mathbb{P}^N$). Then we aim to construct a space of orbits. Two phenomenum occur: one need to exclude some orbits (called unstable) and to identify other ones.

We will mainly study two results: The first, the Hilbert-Mumford theorem is the main tool to understand unstability. The second, Kempt-Ness theorem is a bridge between GIT and Hamiltonian actions of compact Lie groups (via moment map).

We plan to present examples from several contexts: Linear Algebra (Horn's problem), Representation Theory, Representation of quivers, Algebraic Geometry…

[1] Igor Dolgachev, *Lectures on Invariant Theory*, LNS 296, London Math Society, Cambridge University Press.

[2] Ramo Álejandro Urquijo Novella, *GIT quotients and symplectic reduction: the Kempf–Ness theorem*, https://math.uniandes.edu.co/∼florent/resources/teaching/students /Ramon-Urquijo—Kempf-Ness.pdf

[3] Mumford-Fogarty-Kirwan, *Geometric Invariant Theory*, Springer Verlag.

2.2 Symplectic capacities (24h)

Marco Mazzucchelli

One of the earliest results of symplectic topology is the celebrated non-squeezing theorem of Gromov from 1985: the ball $B^{2n}(r)$ admits a symplectic embedding into the cylinder $Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2}$ if and only if the radii satisfy $r < R$. These radii have the following dynamical interpretation. For any Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ with regular level set $H^{-1}(0) = \partial B^{2n}(r)$, the minimal action of the periodic orbits of the Hamiltonian flow $\phi^t_H|_{\partial B^{2n}(r)}$ is precisely πr^2; analogously, if $H^{-1}(0) = \partial Z^{2n}(R)$, the minimal action is πR^2. This suggests that the rigidity asserted by Gromov's theorem is related to the periodic orbits on the boundary of the considered domains.

Inspired by Gromov's theorem and by its interplay with Hamiltonian dynamics, Ekeland and Hofer introduced the notion of symplectic capacity, and constructed the first examples of them. A capacity c is a symplectic invariant that measures the "size", in a suitable sense, of domains contained in symplectic manifolds. Gromov's theorem is a direct consequence of the fact that $c(B^{2n}(r)) = c(Z^{2n}(R))$ for any such capacity c.

In this course, after introducing the needed background, we explore several constructions of symplectic capacities, their applications, and some open questions, including:

- Rigidity phenomena: Gromov non-squeezing, the symplectic camel theorem, and Eliashberg-Kim-Polterovich contact non-squeezing.
- Dynamical applications, and in particular the proof of the Weinstein conjecture for contact-type hypersurfaces of $\mathbb{R}^{2n}$: any such hypersurface admits a closed orbit for its Reeb flow.

- The Viterbo conjecture, asserting that the centrally symmetric convex domains of volume one with the largest symplectic capacities are symplectomorphic to round balls, and its relation with the Mahler conjecture from convex geometry.

References

[1] A. Abbondandolo, B. Bramham, U. Hryniewicz, P. Salomão. *Sharp systolic inequalities for Reeb flows on the three-sphere.* Invent. Math. 211 (2018), no. 2, 687–778.

[2] S. Artstein-Avidan, R. Karasev, Y. Ostrover. *From symplectic measurements to the Mahler conjecture.* Duke Math. J. 163 (2014), no. 11, 2003–2022.

[3] H. Hofer, E. Zehnder. *Symplectic invariants and Hamiltonian dynamics.* Birkhäuser Advanced Texts, 1994. xiv+341 pp.

2.3 Hermite–Einstein metrics and slope stability (24h)

Eveline Legendre

The Kobayashi-Hitchin (KH) correspondence between slope-stable bundles and Hermite-Einstein connections has had an enormous influence on contemporary differential and algebraic geometry. Established on Riemann surfaces in the 1960's by Narasimhan and Seshadri and then in various degrees of generality in the 1980's by leaders like Donaldson, Uhlenbeck and Yau. This result provides a necessary and sufficient algebraic condition (*slope-stability*) for the existence of a solution (*Hermitian Yang–Mills connection/Hermite–Einstein metric*) of certain non-linear partial differential equations. More precisely, given a holomorphic bundle E over a Kähler manifold (X,ω) and denoting the Kähler class $[\omega]\in H^{1,1}_{Dolb}(X)$, the KH correspondence says that

$$E \text{ is slope stable w.r.t } [\omega] \quad \Leftrightarrow \quad \exists! \text{ Hermite–Einstein metric } h \text{ on } E$$

where slope stability is a numerical condition on the slope of all coherent subsheaves of the sheaf of sections of E and, denoting F_h the curvature of the Chern connection of the Hermitian metric h, the Hermite–Einstein condition is the PDE $\mathrm{tr}_\omega F_h = \lambda_E Id_E$, where λ_E is a topological constant.

The KH correspondence provides a Riemannian structure on the moduli space of slope-stable bundles whose geometry is a topological/algebraic invariant of the underlying variety. A particularly striking application of this, is the use of Donaldson Theory to give non-diffeomorphism results for algebraic surfaces by comparing moduli spaces of holomorphic vector bundles and instanton moduli spaces.

Generalisations of the KH correspondence, analogous theories and conjectures have emerged in recent decades, giving many different notions of *stability* depending on the algebraic structure involved (Chow stability, Bridgeland stability, K-stability, weighted K-stability, valuative stability...) and a candidate for a corresponding solution of a geometric PDE (Kähler-Einstein (KE), constant scalar curvature, extremal or weighted extremal metrics, deformed Hermitian Yang–Mills connections, special Lagrangian...). Most of these conjectural pictures are still to be confirmed.

The goal of this course is to introduce the students to the KH correspondence. This necessites to cover the following topics :

- Hermitian connections on holomorphic vector bundles (of any finite rank), Chern-Weyl Theory.
- Bochner-Kodaira-Nakano identity and annulation's theorem.
- Hermite-Einstein metrics and the gauge group action on hermitian connections.
- Slope-stability of sheaves.

References

[1] Hirzebruch, *Topological Methods in Algebraic Geometry* Springer-Verlag 1966.

[2] S.Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, 1987.