Master 2 “Mathématiques Avancées” 2024-2025

Partial Differential Equations and Applications

This program aims to prepare students for research in the field of theoretical and numerical analysis of problems involving partial differential equations (PDEs). It has three components:

1. Refresher Courses in the first 2 weeks aimed at ensuring a common knowledge base for students from various mathematical backgrounds.
   These courses are optional but very strongly advised.

2. Three Basic Courses which offer a broad introduction to the analysis techniques of a large class of partial differential equations.

3. Four Advanced Courses on subjects closely related to current research: optimal transport, the non-linear Schrödinger equation, kinetic theory, stochastic reaction-diffusion equations.

The advanced courses will particularly welcome the participation of PhD students and colleagues.

1 Refresher Courses

Basic tools of functional analysis, Pierre-Damien Thizy (16h)

1. Duality: Hahn-Banach theorem, weak and weak-* topologies, Lebesgue spaces;
2. Distributions: weak derivatives, convolution, fundamental solutions of differential operators;
3. Fourier transform;
4. Sobolev spaces: embeddings, extension and traces, compactness;
5. Weak solutions of PDEs;
6. Spectral analysis in Hilbert spaces.

Starting with PDEs, Alexandre Lanar (16h)

1. Introduction: classifications of PDEs, symbols, notions of solutions.
2. The Laplace equation and second order elliptic operators.
3. The heat equation and second order parabolic operators.
4. Hyperbolic operators.
5. Semigroup theory and applications.

Stochastic tools, Thomas Budzinski (15h)

1. Discrete time martingales: stopping theorems and convergence. Extensions for continuous time martingales.
2. Construction of Brownian motion. Regularity of trajectories.
3. Some properties of Brownian trajectories. Connection with the heat equation.

2 Basic Courses

Quelques modèles et méthodes en sciences du vivant, Thomas Lepoutre (24h)

La modélisation en sciences du vivant est très vaste et nous nous intéresserons ici à des modèles de **dynamique de population** sous formes d'EDP linéaires ou nonlinéaires permettant d'illustrer certains comportements importants. Le cours abordera des équations (ou des systèmes) paraboliques nonlinéaires (qui permettront de s'intéresser à des outils pour l'existence ou l'étude du comportement en temps long). Des modèles de type intégrodifférentiels seront également abordés dans le cadre des populations structurées (cadre où la population est décrite comme une densité d'individus distingués par une caractéristique comme l'âge, la taille ou un trait phénotypique). Dans ce cadre, nous nous étudierons en particulier le comportement des problèmes linéaires.

Nous chercherons dans ce cours à illustrer les différents régimes possibles (dépendant potentiellement d'un paramètre) pour un même modèle (explosion en temps fini, convergence vers un équilibre ou alignement vers un profil) et les outils pour les étudier.

English version: A few models and methods for life sciences

Modeling in the life sciences is a broad topic, and we focus on **population dynamics** in the form of linear or nonlinear PDEs in order to illustrate certain important behaviors. Nonlinear parabolic equations (or systems) will be covered (providing tools for the existence or study of long-time behavior). We will study integrodifferential models in the context of structured populations (where the population is described as a density of individuals distinguished by a characteristic such as age, size or phenotypic trait). In this context, we will study in particular the behavior of linear problems.

In this course, we will illustrate the different possible regimes (potentially depending on a parameter) for the same model (finite-time explosion, convergence towards an equilibrium or alignment towards a profile) and the tools for studying them.

Evolutionary PDEs, Dragos Iftimie (24h)

1. Some properties and reminders of distributions.
2. The Cauchy problem for linear PDEs.
   (a) Variable coefficients. Cauchy-Kovalevskaya theorem, characteristic hypersurfaces and Holmgren's uniqueness theorem. Well-posed problems.
   (b) Constant coefficients.
      - Existence of an elementary solution, the Malgrange-Ehrenpreis theorem. Examples. Necessary and sufficient conditions for hypoellipticity.
      - Local resolubility of the Cauchy problem. Hyperbolicity. Gårding's theorem. Necessary and sufficient conditions for hyperbolicity.
3. Dispersive PDEs.
   (a) A few linear dispersive PDEs and their explicit solutions.
   (b) Non linear Schrödinger equation. Strichartz estimates and some well-posedness results for the Cauchy problem.

4. Symmetric hyperbolic quasilinear systems. Incompressible Euler equations. H^3 solutions and the Beale-Kato-Majda blow-up criterion.

5. Incompressible Navier-Stokes equations. Leray solutions. Uniqueness for small data in dimension 3.

Calculus of variations and elliptic equations, Filippo Santambrogio (24h)

The course will be mainly devoted to the study of the minimizers of integral functionals, their existence, their regularity, and their characterization in terms of solutions of some partial differential equations, but regularity results for the equations themselves will also be presented for their own interest.
The course will be roughly structured into 10 classes as follows:

1. *Introduction and 1D* examples of 1D variational problems (geodesics, brachistochrone, economical growth models) and their applications, tools for existence, Euler-Lagrange equation.

2. *Higher-dimensional calculus of variations and the example of harmonic functions and distributions* Euler-Lagrange in higher dimension, main properties of the solutions of $\Delta u = 0$ in connection with the minimization of the Dirichlet energy

3. *Convexity and semicontinuity* conditions to ensure the semicontinuity for the weak Sobolev convergence of integral functionals and applications to existence results. Notions of convex analysis (Fenchel-Legendre transforms, subdifferentials…).

4. *Convex duality* duality for some "simple" convex variational problems.

5. *Regularity via duality* application of convex duality to some H^1 regularity results.

6. *L^p estimates for the Poisson equation*. Proof by interpolation of the result $\Delta u = f$, $f \in L^p \Rightarrow u \in W^{2,p}$.

7. *Hölder regularity with smooth coefficients*. Morrey-Campanato spaces and applications to the result $\nabla \cdot (a(x)\nabla u) = \nabla \cdot F$, $a, F \in C^{k,\alpha} \Rightarrow u \in C^{k+1,\alpha}$.

8. *Hölder regularity with bounded coefficients*. Proof by Moser's iterations of the De Giorgi regularity result $\nabla \cdot (a(x)\nabla u) = 0$, a bounded and uniformly elliptic but not smooth $\Rightarrow u \in C^{0,\alpha}$ and applications to the solution of the 19th Hilbert problem.

9. *$\Gamma-$convergence and examples*. The general theory of the $\Gamma-$convergence for the limits of variational problems and some example, in particular the optimal quantization of measures (aka optimal location problem).

10. *BV functions, perimeters, and the Modica-Mortola functional*. Few words about the space BV and its role in defining sets of finite perimeter. Proof of the $\Gamma-$convergence of the functionals $\int \varepsilon|\nabla u|^2 + \varepsilon^{-1}W(u)$ towards the perimeter functional.

The knowledge of some functional analysis (in particular, compactness for weak-* convergence and Sobolev spaces) and some measure theory is the main prerequisite for the course.

3 Advanced Courses

Optimal transport: introduction and overview, Cédric Villani (18h)

Born in the late eighteenth century, the field of optimal transport has been revolutionized in the first two decades of the 21th century. At the crossroad of analysis, optimization, partial differential equations and statistics, it has become a classical tool in many fields, from fluid mechanics to non-Euclidean geometry to artificial intelligence. The lines of the course will be:

1. Basic theory, Monge-Kantorovich duality
2. Optimal transport and geometry and curvature
3. A selection of applications and current issues

References :

- C. VILLANI, Optimal transport, old and new, *Grundlehren der mathematischen Wissenschaften*, Vol. 108, Springer, 2008.
- C. VILLANI, Topics in Optimal Transportation, *Graduate studies in mathematics*, Vol. 58, American Mathematical Society, 2003
- F. SANTAMBROGIO , Optimal transport for applied mathematicians, *Progress in Nonlinear Differential Equations and Their Applications*, Vol. 87, Birkhäuser, 2016

Systems of Reaction-diffusion equations: global existence and stochastic modelling , Julien Vovelle (18h)

- Tools for the study of mean-field limits of stochastic systems (Markov processes, Martingales, Propagation of chaos)
- Quadratic systems of reaction-diffusion equations with tamed non-linearity.
- Quadratic systems of reaction-diffusion equations: global existence of smooth solutions.
- Mean-field limits of the stochastic description of bimolecular chemical reactions.

References

- Limites de champ moyen, cours de dea 2001-2002, Cédric Villani.
- Quantitative propagation of chaos in the bimolecular reaction-diffusion model, Lim-Lu-Nolen, 2020
- Global classical solutions to quadratic systems with mass control in arbitrary dimensions, Fellner-Morgan-Tang, 2020

On the non linear Schrödinger equation, Nikolay Tzvetkov (18h)

We will consider the defocusing non linear Schrödinger equation. We will first show that in dimensions ≤ 3 this equations are globally well-posed in various geometric settings. As a by product , we will obtain that the H^1 norm of the solutions are bounded in time. We will then consider the question of the behaviour of the H^s, $s > 1$ norms of the solutions. This question is closely related to the possible migration of the Fourier modes of the solutions from low to high frequencies. We will show that when the problem is posed on the euclidean space $\mathbb{R}^3$ then the H^s, $s > 1$ norms remain bounded in time which prevents the possible migration to high frequencies of the Fourier modes of the solutions. In sharp contrast, we will show that when the problem is posed on the product space $\mathbb{T}^2 \times \mathbb{R}$ then the H^s, $s > 1$ norms of the solutions may be unbounded when the time evolves and thus the migration to higher modes may indeed occur. This phenomenon is some times referenced as a weak wave turbulence.

Plan of the course :

1. Dispersive estimates for the linear equation.
2. Global well-posedness in the energy space.
3. Large data scattering for NLS on $\mathbb{R}^3$.
4. The modified scattering.
5. The resonant system in the periodic setting and its large time analysis.
6. Solutions with unbounded Sobolev orbits for NLS on $\mathbb{T}^2 \times \mathbb{R}$.

Semiclassical dynamics, Laurent Laflèche (18h)

This course will present mathematical tools to describe the links between quantum and classical theories. Classical dynamics of particles are indeed given by Newton laws, and when the number of particles is large, by partial differential equations for continuous distributions, such as kinetic and fluid equations. On the other hand, quantum mechanics is expressed in terms of complex valued wave functions which verify Schrödinger equations.

More generally, semiclassical analysis aims to understand asymptotic expansions in terms of a small parameter often corresponding to the Planck constant. Such expansions usually first require some regularity independent of the small parameter.

The main topics of the lectures will be

- the basis of the underlying physical theories
- the Wigner and Husimi transforms, the Weyl and Wick quantizations
- operator theory, Schatten spaces, trace and semiclassical inequalities
- Quantum optimal transport and Sobolev spaces
- the large particle number approximation and the limit from the Hartree to the Vlasov equation

Some references:

- F. Golse. Mean Field Kinetic Equations - M2 Course Notes. Ecole Polytechnique, 2013. http://www.cmls.polytechnique.fr/perso/golse/M2/PolyKinetic.pdf
- B. Simon. Trace Ideals and Their Applications: Second Edition, volume 120 of Mathematical Surveys and Monographs. American Mathematical Society, 2 edition edition, 2005.
- R. L. Frank, The Lieb-Thirring Inequalities: Recent Results and Open Problems. arXiv:2007.09326, 18 juillet 2020. http://arxiv.org/abs/2007.09326.