Higher Algebra and Formalised Mathematics

Proposal for Advanced Mathematics second year program 2024–2025

Coordinated by
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Lecturers
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Presentation of the Program

Higher algebra, categorification in algebra and geometry, rewriting theory, type theory in computer science and mathematics and formalisation of mathematics have led to recent developments of original and interesting mathematics. This program aims to present a flavour of all these new approaches according to three axes: 1/ categorification and algebraic rewriting, 2/ type theory, 3/ formalised mathematics using proof assistants. The objective is also to prepare students for research in these three very active areas of mathematics at the interplay with fundamental computer science.

The program consists in six courses: three basic courses at the fall semester and three advanced courses during winter (January – March). The basic and advanced courses last 24 hours. Basic courses offer a broad introduction to these three area :

- Algebraic rewriting and categorification;
- Foundations of formalised mathematics and higher-dimensional rewriting;
- Introduction to L∃∀N.

In the advanced courses, the spectrum is deepened in all three areas, while maintaining links between the addressed topics. The themes covered in these courses will be directly related to current research:

- Rewriting theory of higher algebras;
- Calculus of Inductive Constructions and Coq;
- Advanced projects on L∃∀N.

The multiplicity of approaches and concepts around categorification, rewriting, type theory and formalisation of mathematics developed in this course will be carried on in the possibilities of coaching Master 2 thesis in a wide variety of fields. The possibilities for further training after the program are both academic and non-academic, with a buoyant job market in companies for profiles in applied fundamental mathematics.

Basic courses

Algebraic rewriting and Categorification (24 h) - Stéphane Gaussent and Philippe Malbos.

Abstract. Rewriting in algebraic structures is motivated by several issues: to solve the word problem in monoids, to compute ideals in algebras and operads, to solve linear systems of partial differential equations, but also to compute with terms in Lawvere theories or string diagrams in linear monoidal categories. The first part of this course consists of an introduction to the theory of rewriting in a unified framework and to methods of local analysis of confluence and algebraic coherence, the calculation of resolutions and proofs of koszulity and Poincaré–Birkhoff–Witt bases.

The second part presents categorification as a way of revealing hidden structure in well-known algebras and their representations. We will discuss the general topic, and illustrate this theory with the examples of Heisenberg categories and Kac–Moody 2-categories. These categories help us to better understand the representation theory of the symmetric groups, Iwahori–Hecke algebras, Heisenberg algebras, Lie algebras, and quantized enveloping algebras.

Finally, we will introduce rewriting theory in the context of higher algebras. We will then be able to rewrite in monoidal categories of diagrams.

Tentative plan

- Abstract et algebraic rewriting theory
- Higher-dimensional rewriting theory
- Strict monoidal categories and string diagrams
- Categorification
- Heisenberg categories
- Kac–Moody 2-categories

References

- Anna Beliakova & Aaron D. Lauda, *Categorification and Higher Representation Theory*, Contemporary Mathematics, Volume 683 (2017), 361 pages
- Brundan, *On the definition of Kac–Moody 2-category*, Math. Ann. 364 (2016), 353–372
- Alistair Savage, *String diagrams and categorification*, preprint 2018, https://arxiv.org/abs/1806.06873
- Dimitri Ara, Albert Burroni, Yves Guiraud, Philippe Malbos, François Métayer, Samuel Mimram, Samuel, Polygraphs: from Rewriting to Higher Categories, London Mathematical Society Lecture Note Series, 680 pages, to appear, 2024.

Foundations of formalised mathematics and higher-dimensional rewriting (24 h) - Georg Struth.

Abstract. The aim of this course is twofold: Its first part introduces some foundations of interactive theorem proving, namely type theory and the formal proof systems needed for working with the three

proof assistants introduced in this program: Coq, Isabelle/HOL and Lean. Type theory not only provides an alternative to the traditional foundations of mathematics in set theory, it is also strong related to category theory and the functional programming languages on which Coq, Isabelle/HOL and Lean are built. In addition, this first part features a brief overview of the Isabelle/HOL proof assistant, which is based on classical typed higher-order logic. It surveys Isabelle's language for formalising mathematics, its reasoning tools and its mechanisms for engineering mathematical hierarchies.

The second part of this course discusses case studies in formalised mathematics, most of which are motivated by coherence proofs in higher rewriting. Traditionally, rewriting techniques are used for solving word problems in algebra, in algorithms used by computer algebra systems or for evaluating expressions in programs, automated theorem provers and proof assistants. Higher-dimensional rewriting aims at extending such applications to categorical algebra. Here we introduce standard notions of traditional rewriting and the basic categorical machinery of higher rewriting together with higher algebras in which proofs of standard rewriting theorems can be performed. We also show how such algebras can be formalised with Isabelle/HOL and how some key statements from rewriting, including Church–Rosser theorems and variants of Newman's lemma, can be formalised.

Tentative plan

- Type theory, including simple and dependent types, type universes and families, data types, propositions as types;
- Isabelle/HOL, including programming and proving with Isabelle, proof tactics, proof automation and structured proof, type classes and locales;
- Rewriting and higher rewriting, including concepts from abstract, string and term rewriting, polygraphs, strict ω-categories and cubical ω-categories, concepts of higher abstract rewriting;
- Algebras for rewriting and higher rewriting, including Kleene algebras, quantales and their higher variants;
- Cases studies in building mathematical components for classical and higher rewrite proofs with Isabelle/HOL.

References

- Homotopy Type Theory: Univalent Foundations of Mathematics, The Univalent Foundations Program, 2013, http://homotopytypetheory.org/book/
- Philippe Malbos. Lectures on Algebraic Rewriting. Doctoral lecture notes, hal-02461874, 2019.
- Isabelle/HOL: https://isabelle.in.tum.de
- Cameron Calk, Eric Goubault, Philippe Malbos, Georg Struth: Algebraic coherent confluence and higher globular Kleene algebras. Logical Methods Computer Science 18(4), 2022
- Cameron Calk, Philippe Malbos, Damien Pous, Georg Struth: Higher Catoids, Higher Quantales and their Correspondences. CoRR abs/2307.0925, 2023

Introduction to L∃∀N (24 h) - Sophie Morel, Filippo A. E. Nuccio and Xavier-François Roblot.

Abstract. Proof assistants are software that can verify the logical coherence of proofs — or of programs. Although the first proofs assistant are some 50 years old, it is only recently, say in the last two decades at most, that they have been used to check correctness of mathematical arguments. We will introduce Lean-4, a modern proof assistant based on the Calculus of Inductive Constructions that is gaining more and more popularity amongst mathematicians and that has allowed formalisation of strikingly deep results in the last years. The two courses will allow students to move their first steps in the world of mathematical formalisation in Lean-4 and to get acquainted with the challenges and the possibilities offered by this new branch of mathematics, while working on some concrete examples and applying the knowledge acquired in the courses on type theory.

Coding well-known mathematical concepts (in our case, definitions or proofs) requires a good knowledge of the syntax and of the operations allowed in Lean-4, as well as a basic understanding of theoretical underlying architecture. In this first course will discuss:

- the basic interaction with Lean-4: numbers, elementary commands, first tactics;
- the role played by the "proposition as types" point of view;
- more advanced tactics and structured proofs;
- how to browse and use mathlib;
- first elements about translating mathematical concepts in Lean-4: structures, classes, instances.

Alongside classical lectures, students will be required to write their code and to practice the formalisation of small results. Therefore, training sessions will be proposed and a substantial part of the course will be devoted to the development of programming and formalisation skills.

References

- Jeremy Avigad, Leonardo de Moura and Soonho Kong, *Theorem Proving in Lean - Release 3.23.0*, October 2021. *Available at* https://leanprover.github.io/theorem_proving_in_lean/
- Jeremy Avigad and Patrick Massot, *Mathematics in Lean*, October 2023. *Available at* https://leanprover-community.github.io/mathematics_in_lean/index.html

Advanced courses

Rewriting theory of higher algebras (24 h) - Stéphane Gaussent and Philippe Malbos.

Abstract. This course takes a more in-depth look at the notions of rewriting and categorification seen in the first semester, in the basic courses.

Rewriting in higher dimensional categories gives a powerful and intuitive set of tools for studying higher representation theory. It also leads to connections between representation theory and other fields of mathematics, such as low dimensional topology, knot theory, algebraic combinatorics, and mathematical physics.

We will begin by presenting the most recent results in 2-category rewriting, such as rewriting modulo a part of the structure. We will then show how this type of rewriting can be used to obtain normal form results in morphism spaces of certain 2-categories.

Finally, we will introduce rewriting theory in another instance of higher algebras. We will then be able to rewrite in linear monoidal categories of diagrams and Cartesian or linear operads. To this end, we will present advanced theories of algebraic rewriting: rewriting modulo, linear rewriting, layered algebraic rewriting, decreasing rewriting.

Tentative plan

- Rewriting with string diagrams
- Rewriting modulo and linear rewriting
- Layered rewriting in monoidal categories and operads

References

- B. Dupont *Rewriting modulo isotopies in Khovanov–Lauda–Rouquier's categorification of quantum groups*, Advances in Mathematics, Volume 378, Feb 2021
- Y. Guiraud & P. Malbos, *Polygraphs of finite derivation type*, Mathematical Structures in Computer Science, Vol. 28, Issue 2, pp. 155-201, 2018

Calculus of Inductive Constructions and Coq (24 h) - Damien Pous.

Abstract. This is an advanced course designed to provide a comprehensive understanding of the principles and applications of the Calculus of Inductive Constructions (CIC) and the Coq proof assistant. This course explores the intersection of mathematics, computer science, and formal logic.

Tentative plan

- Introduction to CIC:
  - Understanding the fundamentals and the logical foundations of the Calculus of Inductive Constructions.
  - Dependent types and their role in CIC.
- Coq Overview and Proof Development
  - Introduction to the Coq proof assistant, basic commands and tactics.
  - Formal verification and formal proofs in Coq.
  - Structural induction and recursion.
  - Proving theorems and properties of mathematical structures.
- Advanced Topics:
  - Inductive families, dependent patterns, and universe hierarchies.
  - Handling computational and non-computational proofs.

References

- Yves Bertot and Pierre Castéran, *Interactive Theorem Proving and Program Development* Coq'Art: The Calculus of Inductive Constructions. Springer-Nature, 2004.

Advanced project on L∃∀N (24 h) - Sophie Morel, Filippo A. E. Nuccio and Xavier-François Roblot.

Abstract. The main goal of this second course is to acquire deeper knowledge and understanding of Lean-4 and to discuss the formalisation of more advanced material. As in the first course, practical sessions and theoretical classes will be alternated so that students will have time to test on their own the concepts explained during lectures and to develop the formalisation of some pieces of interesting mathematics. The courses can focus either on how to use and develop more structured mathematical hierarchies or on how to design and implement more advanced Lean-4 programs and tactics. The precise choice of the topics that will be covered will depend on the participants' preferences and knowledge.

References

- *Homotopy Type Theory*, The Univalent Foundations Program, IAS (2013). *Available at* https://homotopytypetheory.org/book/
- The mathlib Community *The lean mathematical library*. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs* (*CPP 2020*). Association for Computing Machinery, New York, NY, USA, 367–381 (2020). DOI: https://doi.org/10.1145/3372885.3373824
- David K. Christiansen, *Functional programming in Lean*, Microsoft Corporation, 2023. *Available at* https://lean-lang.org/functional_programming_in_lean/title.html