Representation Theory, from Schubert Calculus to Rewriting

Representation theory is a branch of mathematics where other fields, like combinatorics, geometry and category theory interplay to give new results and insights. In the last years, new approches to representation theory have emerged, mainly from the categorical and the geometry points of view. On the category side, categories with tensor product, called monoidal categories, like the category of vector spaces, play nowadays a major role in representation theory.

To cite some of them, let us mention the category of Soergel's bimodules, the Heisenberg category or the 2-category of Kac-Moody (a generalization of a monoidal category). These linear 2-categories categorify classical objects of representation theory, such as Iwahori-Hecke algebras, Heisenberg algebras, quantum groups associated with Kac-Moody algebra. The transition from an algebra to a monoidal category, or to a 2-category, enriches the structure and makes it possible to obtain positive results and bases of these algebras, or even their representations.

These categories have presentations by generators and relations that can be expressed, in a graphical way, in terms of string or diagram calculus. These diagrams are graphs drawn in a horizontal band of $\mathbb{R}^2$ verifying certain properties. They can be composed horizontally by juxtaposing graphs, but also vertically, by concatenation, and one can make linear combinations of them.

The study of these linear 2-categories and their presentations requires the implementation of advanced combinatorial tools, such as Coxeter systems, symmetric functions or Kac-Moody root systems. The rewriting practiced in the context of polygraphs also makes it possible to study these presentations from a constructive and algorithmic point of view, making it possible to calculate invariants of these 2-categories, and bases of the spaces of morphisms.

The objective of this course is to present to students a flavour of all these new approaches according to three axes: Schubert calculus, diagrammatic categories and rewriting. The course on rewriting will develop the combinatorial study of the presentations of the 2-categories. The two fundamental courses of representation theory and Schubert calculus will focus on the group $\mathsf{GL}_n(\mathbb{C})$: on the one hand with the study of its representations, pushing to the construction of the Gelfand-Tsetlin bases, on the other hand, with the combinatoric of Schubert calculus where the symmetric group and symmetric functions are essential tools.

In the advanced courses, the spectrum widens in all three areas, while maintaining links between the three axes. Thus the advanced combinatorics course will deal with Coxeter systems and harmonic polynomials, while on the side of representation theory, diagrammatic categories will appear. These monoidal linear categories and their generalizations are studied from a constructive point of view in the advanced course of categorical and operative rewriting.

The multiplicity of approaches and concepts around the theory of representations and rewriting developed in this course will continue in the possibilities of coaching Master 2 thesis students in a wide variety of fields.

Basic Courses (24h)

- *Schubert calculus*, Riccardo Biagioli and Philippe Nadeau

- *Representation Theory of* $\mathrm{GL}_n(\mathbb{C})$, Stéphane Gaussent and Nicolas Ressayre
- *Algebraic Structures of Rewriting*, Philippe Malbos and Fabio Zanasi

Advanced Courses (24h)

- *Coxeter Systems*, Riccardo Biagioli and Olivier Mathieu
- *Diagrammatic Categories and Representations*, Stéphane Gaussent and Kenji Iohara
- *Operadic and categorical rewriting*, Yoann Dabrowski and Philippe Malbos.

The course on rewriting will develop the combinatorial study of the presentations of the 2-categories. The two basic courses of representation theory and Schubert calculus will focus on the group $\mathsf{GL}_n(\mathbb{C})$: on the one hand with the study of its representations, pushing to the construction of the Gelfand-Tsetlin bases, on the other hand, with the combinatoric of Schubert calculus where the symmetric group and symmetric functions are essential tools.

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Schubert calculus

Basic Course, Riccardo Biagioli and Philippe Nadeau

Schubert calculus is the area of enumerative geometry which aims at determining certain intersection numbers between subvarieties in projective space. In these lectures we will focus on the two cases of the Grassmannian variety and the complete flag variety. The decomposition of these spaces in Bruhat cells allows us to give in each case a basis of the cohomology (or the Chow ring) associated with the varieties, and to then use deep combinatorics to tackle the corresponding intersection problems.

These lectures will be mainly focused on these combinatorial aspects. In the Grassmannian case, integer partitions and symmetric polynomials play a fundamental role. The intersection numbers are given by the Littlewood-Richardson coefficients. In the case of the complete flag variety, the Bruhat decomposition is indexed by permutations, and multivariate polynomials are naturally involved. We will focus on several aspects of Schubert polynomials, originally introduced by Lascoux and Schützenberger to represent a basis of the cohomology ring.

Contents

- Partitions, symmetric group, symmetric polynomials.
- The Grassmannian variety, Schur polynomials.
- The complete flag variety.
- Schubert polynomials.

References

- L. Manivel, *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence.* Cours Spécialisés, Soci. Math. France, 1998.
- Fulton, *Young Tableaux.* London Math. Soc. Student Texts 25, 1996.
- Macdonald, *Notes on Schubert polynomials*, Publications du L.A.C.I.M., vol. 6, Université du Québec, Montréal, 1991.

Representation Theory of $\mathrm{GL}_n(\mathbb{C})$

Basic Course, Stéphane Gaussent and Nicolas Ressayre

This lecture is meant to be a first course in the representation theory of reductive algebraic groups and their Lie algebras. But to avoid structure theorems of these groups, we will focus on the typical example of the general linear group $\mathsf{GL}_n(\mathbb{C})$ of $n \times n$ matrices with nonvanishing determinant.

After introducing the finite dimensional complex representations of $\mathsf{GL}_n(\mathbb{C})$, we will study some combinatorial aspects of the theory like the Littlewood-Richardson rule, the Weyl character formula, the Gelfand-Tsetlin bases and the Littelmann paths model. Some of these topics generalise to the representation theory of reductive groups but we will not deal with this general case in this lecture.

Unfortunately, even though all the topics addressed in this course are quite accessible, they do not appear all together in one bibliographical source. Furthermore, there is no accessible textbook on the Gelfand-Tsetlin bases nor on the Littelmann path model. All definitions and results, concerning those, will be provided in this lecture.

References

- W. Fulton and J. Harris, *Representation theory. A first Course*, Grad. Texts in Math. 129, Springer, 1991.
- T. Bröcker, and T. tom Dieck, *Representations of compact Lie groups*, Grad. Texts in Math. 98, Springer, 1985.
- O. Mathieu, *Le modèle des chemins (d'après P. Littelmann)*, Séminaire Bourbaki 1994-1995, n^o 798, Astérisque 237, Soc. Math. France, 209–224.
- D. Bump, and A. Schilling, *Crystal bases, Representations and combinatorics*, World Scientific, 2017.

Algebraic Structures of Rewriting

Basic Course, Philippe Malbos and Fabio Zanasi

Rewriting in algebraic structures has appeared independently in several situations: to decide the word problem in monoids, to compute within ideals in algebras and operads with the Gröbner bases, to solve linear systems of partial differential equations, but also in Lawvere theories and in monoidal categories for some applications in fundamental computer science, through term rewriting.

The first part of this course consists of an introduction to the rewriting theory in a unified framework and to methods of local analysis of confluence and consistency. Procedures of homotopic completion-reduction and rewriting modulo will also be addressed.

In a second part, we will study two fields of applications: First, linear rewriting, constituting a computational model in associative algebras, with local confluence analysis methods by the Janet-Shirshov-Buchberger criterion, the calculation of resolutions, some proofs of koszulity and Poincaré-Birkhoff-Witt bases; Second, rewriting in monoidal categories, allowing to explain computer calculation models such as quantum processes, competing systems and Petri's lattices.

This course is designed to be accessible to both students of this course and of the Master 2 in fundamental computer science from ENSL.

References

- W. W. Adams and P. Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Math, vol. 3, Amer. Math. Soc., 1994.
- D. J. Anick, On the homology of associative algebras, Trans. Amer. Math. Soc., 296 (1986), 641–659.
- T. Mora, An introduction to commutative and noncommutative Gröbner bases, Theoretical Computer Science, Vol. 134 (1994), 131–173.
- Terese, *Term Rewriting Systems*, Cambridge tracts in theoretical computer science, 55, 2003.

Coxeter Systems

Advanced Course, Riccardo Biagioli and Olivier Mathieu

This course will start with the classification of finite groups generated by reflections in the euclidean space. Some geometric notions will be introduced as : roots systems, hyperplanes arrangements, alcoves and Weyl chambers. We will follow the book of Humphreys.

All this will bring us to the definition of Coxeter groups via generators and relations. We will study the combinatorial properties of length function, reduced decompositions, strong and weak order. The emphasis will be on the combinatorial and enumerative questions raised by such notions.

We will introduce also the Kazhdan–Lusztig polynomials that play important roles in various aspects of the representation theory of reductive algebraic groups, and in the cohomology of the Schubert varieties. We will be mainly interested in showing some explicit combinatorial interpretations for such polynomials and the related family of R-polynomials. This part will be based on the book of Björner and Brenti.

Finally, generalizing the Schubert calculus in the cohomology, we will introduce the Schubert calculus in the K-theory of the complete flag variety, as well as, its connections with the harmonic polynomials.

References

- A. Brenti and F. Björner, *Combinatorics of Coxeter groups*, Grad. Texts in Math. 231, Springer, 2005.
- J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge studies in advanced mathematics, 29, 1990.

Diagrammatic Categories and Representations

Advanced Course, Stéphane Gaussent and Kenji Iohara

The monoidal category of webs allows to give a presentation by generators and relations of a category of some representations of the quantum group associated to the Lie algebra $\mathfrak{sl}_n$. In a first part, the course will study this presentation in terms of the rewriting tools that have been developed in the third fundamental lecture on rewriting theory. Further, works of Fontaine, Kamnitzer and Kuperberg relate this category to the geometry of the associated affine grassmannian and affine building.

Some other examples of categories presented by diagrams will be then discussed. In particular, the course will focus on the group algebras of Artin-Tits groups and their quotients that give the Iwahori-Hecke algebras of type A and B and their affinizations.

References

- S. Cautis, J. Kamnitzer, C. Morrisson, Webs and quantum skew Howe duality, Mathematische Annalen 360 (2014), 351–390
- B. Fontaine, J. Kamnitzer, G. Kuperberg, Buildings, spiders, and geometric Satake, Compositio Mathematica
- J. J. Graham and G. I. Lehrer, The representation theory of affine Temperley-Lieb algebras, Enseign. Math. (2) 44 (1998), 175–218.

Operadic and categorical rewriting

Advanced Course, Yoann Dabrowski and Philippe Malbos

Several constructive homological methods using noncommutative Gröbner bases are known to build free resolutions of associative algebras. These methods allow to link Koszul's property for an associative algebra to the existence of a quadratic Gröbner basis of its ideal of relations.

In this course, we present these constructions in the context of rewriting in higher dimensions allowing to generalize the notion of Gröbner bases to linear monoidal categories. In particular, this generalization makes it possible to go beyond the monomial termination orders. As applications, we will show how to build cofibrant replacements of algebras, calculate linear bases of diagrammatic algebras and prove Koszulity results.

Then, we will study the generalization of Gröbner bases to operads, with an operadic version of the local confluence criteria and the Buchberger algorithm introduced by Dotsenko and Khoroshkin. We will show that these Gröbner bases allow to obtain PBW criteria of Koszulity for symmetric quadratic operads.

This course is designed to be accessible to both students of this course and of the Master 2 in fundamental computer science from ENSL.

References

- V. Dotsenko, A. Khoroshkin, Gröbner bases for operads, Duke Math Journal, 2010.
- Y. Guiraud, E. Hoffbeck, P. Malbos, Convergent presentations and polygraphic resolutions of associative algebras, Mathematische Zeitschrift, 2019
- V. Ginzburg and M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994), no. 1, 203–272.