Probability

This Master 2 program includes:

- a Refresher Course (optional but strongly suggested) of 10 hours by Charles-Edouard Bréhier, that will revise some basic *Stochastic Tools*;

3 Basic Courses (20 hours+6 hours of tutorials)

- Stochastic Calculus, by Grégory Miermont,
- Statistical Physics, by Christophe Garban,
- Random Walks on graphs, by Fabio Toninelli;

and 3 Advanced courses (24 hours)

- High-dimensional phenomena, by Guillaume Aubrun,
- Random graphs, by Dieter Mitsche
- Integrable probability, by Jérémie Bouttier.

This Master program proposes three rather complementary directions.

The courses on "Stochastic Calculus" and "High-dimensional phenomena" give a glimpse of the links between probability and analysis, in particular the concentration of measure phenomenon. They may interest also students attending the PDE Master program.

The course on "Statistical Physics", the one on "Random walks on graphs" and the one on "Random graphs" give an introduction to discrete probability and bring up a variety of research themes such as random geometry, out-of-equilibrium statistical physics and random walks.

Finally, the advanced course on "Integrable probability" has a strong connection both with statistical physics and with combinatorics; it allows to get in touch with extremely active research topics, such as growth models in the KPZ universality class, determinantal processes and the connection between random tilings and the Gaussian Free Field. The courses on "High dimensional phenomena" and "Random graphs" are also aimed at students who plan to pursue a PhD thesis in mathematical statistics.

Stochastic tools

Refresher course, Charles-Edouard Bréhier

This course, that will revise some bases the theory of stochastic processes, will serve as a preparation for the courses on "Stochastic Calculus" and "Random walks on graphs".

Contents :

1. Discrete time martingales; stopping theorems and convergence. Extensions for continuous time martingales.
2. Construction of Brownian Motion. Regularity of trajectories.
3. Some properties of the Brownian trajectories. Connection with the heat equation.

Stochastic Calculus

Basic Course, Grégory Miermontv

This lecture series will present some of the most important tools allowing to build and study continuous-time stochastic processes, the central example of which is of course Brownian motion. To this end, we will have to introduce and study semimartingales, a rich class of processes for which one can develop a differential and integral calculus, and set and solve certain type of stochastic differential equations.

Just as for the familiar ordinary differential equations (or PDEs), the motivation to study such stochastic differential equations comes from the goal of understanding the global behavior of random processes by equations describing their infinitesimal behavior. But since we are dealing with random processes, these equations contain a random "noise", which informally is an infinitesimal increment of Brownian motion. The main problem of their study comes from the fact that Brownian motion (and therefore the other processes of interest) have too rough trajectories (nowhere differentiable, for instance) for the usual differential and integral calculus to make sense.

In front of this obstacle, we will develop a notion of stochastic integral, due to Itô. It will give rise to a particular integral calculus, in which Itô's formula acts as an integration by parts (or a fundamental theorem of analysis) of a new kind. This integral calculus will allow us to study the stochastic differential equations for continuous semimartingales, et will shed a new light on these processes, for instance via Lévy's characterization of Browian motion, or the Dubins-Schwarz theorem according to which continuous martingales are appropriate time-changes of Brownian motion.

Contents :

- Generalities on continuous-time processes
- Continuous-time martingales, regularization. Local martingales, semimartingales. Bracket of a continuous semimartingale.
- Stochastic integration with respect to a continuous semimartingale.
- Itô's formula and applications. The Theorems of Lévy, Dubins-Schwarz, Girsanov.
- Stochastic differential equations.
- Continuous-time Markov processes. Generators. Diffusions.

References :

- Karatzas-Shreve: Brownian motion and stochastic calculus
- Le Gall: Brownian motion and stochastic calculus
- Mörters-Peres: Brownian motion
- Revuz-Yor: Continuous martingales and Brownian motion
- Varadhan: Stochastic processes

Statistical mechanics

Basic Course, Christophe Garban

In statistical physics, one is interested in physical models made of a large number of microscopic elements which interact together in a simple fashion. The goal is then to understand how come such simple microscopic mechanisms can generate interesting (and surprising!) macroscopic phenomena such as phase transitions or symmetry breaking. This program has lead to the development of an important branch of probability theory and the aim of this course is to give a panorama of the field together with tools and techniques that are used in statistical mechanics. We shall focus on three fundamental models: percolation, Ising model and $O(n)$ spherical spin model.

Program of the course

1. Percolation
    - Definition, phase transition
    - FKG inequality, $p_c = 1/2$
    - Exponential decay in the sub-critical regime
    - Russo-Seynour-Welsh theorem for critical percolation
2. Ising model
    - Definition, correlation inequalities
    - Infinite volume limit, Free energy, phase transition
    - Low temperature and Peierls argument
    - Uniqueness at high temperature
3. Phase transition KT (Kosterlitz-Thouless) *Nobel price in phyics 2016*
    - Spin models with continuous symmetry ($\sigma_x \in \mathbb{S}^d$, $d \geq 1$)
    - No symmetry breaking in dimension 2 (Mermin-Wagner theorem)
    - Gaussian Free Field
    - Vortices and Coulomb gas
    - A glimpse of Frölich-Spencer Theorem on the KT transition for the XY model [$\sigma = (\sigma_x)_{x\in\mathbb{Z}^2} \in (\mathbb{S}^1)^{\mathbb{Z}^2}$]

References

- W. Werner, Percolation et modèle d'Ising, Soc. Math. France, 2009.
- Y. Velenik, Introduction aux champs aléatoires markoviens et gibbsiens, `http://www.unige.ch/math/folks/velenik/Cours/2006-2007/Gibbs/gibbs.pdf`.

Random walks on graphs

Basic Course, Fabio Toninelli

A reversible Markov chain can be seen as a random walk on a weighted graph, whose geometry may allow to precisely describe the behavior of the walk, and vice-versa. This course will deal with the study of such walks, and notably the characterization of recurrence and transience, via the electric network properties of the graph. We will explain also the link between these properties and some natural random objects defined on the graph, such as the loop-erased random walk, uniform random spanning trees, the Gaussian Free Field, etc. In the second half of the course, we will discuss the question of the speed of convergence towards the stationary measure for random walks on finite graphs, in terms of mixing time (total-variation convergence) and of spectral gap (L^2 convergence).

Contents:

1. Random walks and electric networks
   - Discrete Dirichlet form, energy
   - Characterization of the asymptotic behavior via the network resistance
   - The case of random walks on trees; examples.
   - Uniform random spanning tree of a graph, matrix-tree theorem, loop-erased random walk
   - Gaussian Free Field on a graph
2. Random walks on finite graphs: speed of convergence
   - Mixing time T_{mix} (reversible and non-reversible random walks)
   - T_{mix} upper bounds: path coupling and weighted distances
   - Spectral gap, variational principle and time correlations
   - Bounding the spectral gap: geometric comparison methods
   - The cutoff phenomenon

References

- R. Lyons and Y. Peres, *Probability on trees and networks*, Cambridge University Press, 2017.
- Y. Le Jan, *Markov paths, loops and fields*, École D'Été de Probabilités de Saint-Flour XXXVIII–2008, Springer Lect. Notes in Math. 2026, 2011.
- D. Levin, Y. Peres, E. Wilmer, *Markov Chains and mixing Times*, Amer. Math. Soc., 2009.

High-dimensional phenomena

Advanced Course, Guillaume Aubrun

This course studies phenomena of high dimension in analysis and probability. We will focus on several aspects of the *concentration of measure* phenomenon: when studying problems involving a large number of variables, it is common that the quantities of interest “concentrate” around a typical value.

This principle is fundamental for the study of asymptotic phenomena (i.e. when dimension goes to infinity) in geometry of finite-dimensional normed spaces. We will discuss in particular the geometry of the Banach–Mazur compactum (the space of normed spaces of fixed dimension), where the probabilistic method plays a fundamental role.

Contents:

- Volume of convex bodies: Brunn–Minkowski inequality and consequences.
- Concentration of measure. Concentration for sums of i.i.d. random variables, concentration on the sphere, on Gaussian space. Connections with isoperimetry. Johnson–Lindenstrauss lemma.
- Covering numbers, connections with error-correcting codes.
- Gaussian processes: Dudley inequality, Sudakov inequality, chaining.
- Geometry of high-dimensional convex bodies: John's theorem, Gluskin's theorem (diameter of the Banach–Mazur compactum).
- Dvoretzky's theorem (almost Euclidean sections of convex bodies) and consequences.
- Grothendieck's inequality ; link with semi-definite programming.

References:

- Roman Vershynin, *High-dimensional Probability*, Cambridge Univ. Press, 2018. Also `https://www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.pdf`
- Guillaume Aubrun et Stanisław Szarek, *Alice and Bob meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory*, Mathematical Surveys and Monographs Volume 223 (2017).
- Shiri Artstein–Avidan, Apostolos Giannopoulos et Vitali Milman, *Asymptotic Geometric Analysis, Part I*, Math. Surveys Monogr. 202, Amer. Math. Soc. 2015.
- D. Li et H. Queffélec, *Introduction à l'étude des espaces de Banach*, Soc. Math. France, 2004. English translation, Cambridge Univ. Press, 2018.

Random graphs

Advanced Course, Dieter Mitsche

In the last years, complex networks have become central elements in many areas (telecommunication networks, internet, neural networks, social networks, propagation of infectious diseases, propagation of rumors, ...). It is a booming area, and it is crucial to develop mathematical models to represent these networks.

A network is often modelled by a random graph, and this course proposes the study of different random graph models, in particular the Erdős-Rényi random graph model, the configuration model and random geometric graphs. A special focus of this course will be given on the threshold of the giant component in different graph models.

Contents:

1. Erdős-Rényi model
   - Introduction, subgraph count
   - Local weak convergence
   - Phase transition, appearance of a giant component
   - Hamiltonicity
2. Configuration model
   - Differential equation method
   - Emergence of a giant component
3. Random geometric graphs
   - Euclidean random geometric graphs - emergence of the giant component
   - Introduction to random hyperbolic graphs

References:

- N. Alon, J. Spencer, The probabilistic method, 3rd ed., John Wiley & Sons, 2008.
- C. Bordenave, Lecture notes on random graphs and combinatorial optimization, `https://www.math.univ-toulouse.fr/~bordenave/coursRG.pdf` [soon to change to Marseille]
- A. Frieze, M. Karonski, Introduction to random graphs, Cambridge University Press, 2015.
- M. Penrose, Random geometric graphs, Oxford Univ. Press, 2003.
- R. van der Hofstad, Random graphs and complex networks, Vol. 1, Cambridge Series in Statistical and Probabilistic Mathematics, 2017. Volume 2, `https://www.win.tue.nl/~rhofstad/NotesRGCN.htmlhttps://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf`

Integrable probability

Advanced Course, Jérémie Bouttier

In 1961, Stanisław Ulam (who is better known for his participation to the Manhattan Project) raised the question of the *longest increasing subsequence*, which can be formulated as follows. For σ a permutation of $\{1, \dots, n\}$, we denote by $\ell_n(\sigma)$ the maximal length of an increasing subsequence of σ. For instance, for $\sigma = (4, 3, 7, 5, 2, 6, 1)$ we have $\ell_n(\sigma) = 3$, as realized for instance by the subsequence $(4, 5, 6)$. What can be said about $\ell_n(\sigma)$ when σ is drawn uniformly at random in the symmetric group of order n, and we let n tend to infinity?

This problem, popularized by Hammersley, led to many works. It is relatively elementary to see that the order of magnitude of $\ell_n(\sigma)$ is $\sqrt{n}$. In 1977, Vershik-Kerov and Logan-Shepp showed that, in fact, $\ell_n(\sigma)/\sqrt{n} \to 2$ in probability. Finally, in 1998, Baik, Deift and Johansson studied the fluctuations of $\ell_n(\sigma)$ around $2\sqrt{n}$ and showed the following remarkably precise result:

$$\mathbb{P}\left(\frac{\ell_n(\sigma) - 2\sqrt{n}}{n^{1/6}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s)$$

where F_2 is the so-called Tracy-Widom distribution, also describing the fluctuations of the largest eigenvalue of a random hermitian matrix.

This unexpected link between a combinatorial problem and random matrix theory is illustrative of the richness of *integrable probability*, a booming subject which also uses methods from mathematical physics, representation theory, functional analysis, etc. The purpose of this course is to give a glimpse of this subject.

Contents

- Combinatorial tools: random partitions and permutations, symmetric functions, Young diagrams and Young tableaux, Robinson-Schensted-Knuth algorithm.
- Analytic tools: determinantal processes, Fredholm determinants, orthogonal polynomials, variational methods, fermionic methods.
- Applications: random tilings, last-passage percolation, growth models, limit shapes, KPZ universality class.

References

1. Dan Romik, *The surprising mathematics of longest increasing subsequences*, Cambridge University Press, 2015. `https://www.math.ucdavis.edu/~romik/book/`

2. Alexei Borodin et Vadim Gorin, *Lectures on integrable probability*, Probability and statistical physics in St. Petersburg. `https://arxiv.org/abs/1212.3351`