Partial Differential Equations, Analysis and Use

This program aims to prepare students for research in the field of theoretical and numerical analysis of problems involving partial differential equations (PDEs). It has three components:

- Refresher Courses in the first 2.5 weeks aimed at ensuring a common knowledge base for students from various mathematical backgrounds. These courses are optional but very strongly advised.
- three Basic Courses which offer a broad introduction to the analysis techniques of large classes of partial differential equations and the discrete approximation of their solutions.
- three Advanced Courses on subjects closely related to current research: the analysis of Navier-Stokes equations in Fluid Mechanics, the use of EDPs from biological sources. and the analysis of these models, and finally, an introduction to statistical mechanics and the origin of gases.

Refresher Courses

- *Basic tools of functional analysis*, Laure Saint-Raymond, 16h
- *Stochastic tools*, Charles-Edouard Bréhier, 10h
- *Starting with PDEs*, Petru Mironescu, 16h

Basic Courses (24h)

- *Variational methods and elliptic PDEs*, Olivier Druet
- *Evolution equations*, Francesco Fanelli
- *Approximation by PDEs*, Julien Vovelle

Advanced Courses (24h)

- *Scaling limits and kinetic theory*, Sergio Simonella
- *Mathematical methods for the Navier-Stokes equations*, Lorenzo Brandolese
- *Analysis of PDE models for biology*, Vincent Calvez.

Refresher courses

Basic tools of functional analysis, Laure Saint-Raymond, 16h

Contents

- Duality : Hahn-Banach theorem, weak and weak* topologies, Lebesgue spaces (3h)
- Distributions : weak derivatives, convolution, regularization (2h)
- Fourier transform (2h)
- Sobolev spaces : embeddings, extension and traces, compactness (3h)
- Weak solution of PDEs : notions of weak solution, a priori estimates, stability (3h)
- Compactness : compensated compactness, averaging lemma, hypoellipticity (3h)

Stochastic tools, Charles-Edouard Bréhier, 10h

- Discrete time martingales; stopping theorems and convergence. Extensions for continuous time martingales.
- Construction of Brownian Motion. Regularity of trajectories.
- Some properties of the Brownian trajectories. Connection with the heat equation.

Starting with PDEs, Petru Mironescu, 16h

Contents

- Geometric aspects of integration theory. Area, co-area, jacobians
- The Bochner integral
- Comparison principles for first and second order partial differential equations. A priori estimates
- Perron's method. A potential theoretic point of view of smoothness
- The energy method. Uniqueness and domain of influence
- Semi-group methods in the study of evolution equations. Existence, smoothness and additional smoothness

Variational methods and elliptic PDEs

Minimal and constant mean curvature surfaces

Advanced course, Olivier Druet

Aim of the course. introduction to classical tools in elliptic PDEs and variational calculus through the study of minimal and constant mean curvature surfaces.

Contents

- Local theory of curves and surfaces.
- Schauder elliptic theory : Plateau's problem for minimal graphs.
- De Giorgi-Nash-Moser elliptic theory : Plateau's problem for parametrized disks. Tools : regularity theory, Sobolev spaces, ...
- Variational problems with lack of compactness : Plateau's problem for CMC surfaces. Tools : compactness by compensation, blow up analysis.

Evolution equations

Advanced course, Francesco Fanelli

Contents

- Introduction

  classification of PDEs, basic definitions

  evolution PDEs: well-posedness, weak and strong solutions

- Transport equations

  characteristics

  Solutions in Lebesgue and Sobolev spaces

  back to characteristics: introduction to Hamilton-Jacobi equations and conservation laws

- Parabolic equations

  heat equation: fundamental solution, maximum principles, energy methods

  the general case: existence of weak solutions, regularity, uniqueness; maximum principles and Harnack inequality

  a non-linear example: the QG equation

- Littlewood-Paley theory

  Littlewood-Paley decomposition

  Besov spaces

  Elements of paradifferential calculus: paraproducts, paralinearization

- Hyperbolic equations and systems

  wave equations with regular coefficients

  first-order hyperbolic Friedrichs symmetrizable systems

  quasi-linear equations and symmetric systems

- Introduction to asymptotic analysis

  back to the QG equation: long-time behaviour

  introduction to the study of fluids in fast rotation

Approximation by PDEs

Advanced course, Julien Vovelle

In this course, we will see how to understand and describe the large scale limit of various discrete evolution systems (random and deterministic) with the help of partial differential equations. This will be the occasion to use and discover some standard tools from the theory of PDEs, of numerical analysis, and of statistical physics.

Contents

- Martingale in continuous time
- Discrete conservation laws, systems of interacting particles and their asymptotic description by PDEs
- Interacting particles systems

  Independent random walk

  Model of random interface
- Discrete conservation laws (Finite Volume method)

  Parabolic equations

  Hyperbolic equations

  The case of the linear transport equation: optimal convergence estimate by two different methods (deterministic/probabilistic).

Scaling limits and kinetic theory

Advanced course, Sergio Simonella

The kinetic theory of rarefied gases, initiated by Boltzmann, realizes the program of the rigorous connection between macroscopic effective equations and interacting models at the atomic scale.

The object of this course is to introduce a basic formalism in statistical mechanics, present the fundamental results of the theory for systems far from equilibrium, and finally the open problems of current interest.

Contents

- Equilibrium systems

  Thermodynamic system

  Statistical ensembles

  Dilute gas, correlation functions

  Thermodynamic limit (notions)

- Kinetic theory

  Scaling limits: Vlasov, Boltzmann, Landau

  Lorentz gas, hard spheres: law of large numbers and Boltzmann equation

  Hierarchical method

  Implications of the Hewitt-Savage theorem

  Irreversibility, entropy

  Central limit theorem, fluctuating Boltzmann equation.

Mathematical methods for the Navier-Stokes equations

Advanced course, Lorenzo Brandolese

This course is devoted to one of the fundamental models of fluid mechanics, the Navier-Stokes equations. We will present several methods for studying such equations (compactness arguments, fixed point methods, dyadic decompositions, Fourier splitting device, bilinear estimates, etc.).

Besides the classical questions of the existence of weak solutions and of unique strong solutions, we will study in some details the problems related to the large time and the far field behavior of the flows.

The last part of the course will be a quick introduction to more complicated models in fluid mechanics, where the equation of the velocity field is coupled with other physical quantities (temperature, magnetic field, etc.).

Contents

- Finite energy solutions

  Leray's weak solutions.

  About the regularity of weak solutions.

  Energy dissipation and Fourier splitting method.

  Asymptotic profiles as $t \to +\infty$.

- Mild solutions

  Kato's solutions and their uniqueness in critical spaces.

  More scale-invariant spaces. Besov spaces and fast oscillating flows.

  The limit case: the Koch-Tataru theorem and norm inflation phenomena.

  Finite time blowup for a Navier-Stokes like equation.

  Self-similar solutions.

- Asymptotic behavior as $|x| \to +\infty$.

  Miyakawa's theorem.

  Point-wise upper and lower bounds.

- Introduction to some other model in fluid mechanics

  Geophysical flows and the Boussinesq approximation. Magnetohydrodynamics.

Analysis of PDE models for biology

Advanced course, Vincent Calvez

The aim of this course is to investigate several Partial Differential Equations models with applications to spatial ecology and evolutionary biology. The common topic will be the description of a population with respect to various structural variables: space, age, phenotypic trait.

The course will address a balance of old and new problems. Focused applications are: species invasion, collective motion, quantitative genetics. Mathematical tools will be borrowed from previous analysis and PDE courses. Specific tools will be also developed during the course.

- Reaction-diffusion equations in spatial ecology
  - Fisher's equation and propagation phenomena

    Existence of traveling waves; spreading speed for the Cauchy problem

    Case study: H. Berestycki et al. Can a Species Keep Pace with a Shifting Climate? Bull. Math. Biol. 71 (2009), 399–429.
  - Kinetic modeling of species propagation

    Existence of traveling waves; spreading speed for the Cauchy problem; wave acceleration

    Case studies: J. Saragosti et al. Directional persistence of chemotactic bacteria in a traveling concentration wave. Proc. Nat. Acad. Sci. of the USA 2011, 16235–16240.

    E. Bouin, V. Calvez and G. Nadin. Propagation in a kinetic reaction-transport equation: traveling waves and accelerating fronts. Arch. Rat. Mech. Anal. 217 (2015), 571–617.
- Reaction-diffusion equations in evolutionary biology
  - Mutation-selection equilibria and concentration phenomena

    Asymptotic analysis in the regime of small variance; convergence towards the viscosity solution of a Hamilton-Jacobi equation; existence and uniqueness

    Case study: G. Barles, S. Mirrahimi, B. Perthame. Concentration in Lotka-Volterra Parabolic or Integral Equations: A General Convergence Result. Methods Appl. Anal. 16 (2009), 321–340.

    V. Calvez and K.-Y. Lam. Uniqueness of the viscosity solution of a constrained Hamilton-Jacobi equation, `https://arxiv.org/abs/1809.05317`, 21 pages.

- Fisher's infinitesimal model

  Asymptotic analysis in the regime of small variance; hydrodynamic limit

  Case study: V. Calvez, J. Garnier and F. Patout, A quantitative genetics model with sexual mode of reproduction in the regime of small variance, `https://arxiv.org/abs/1811.01779`, 35 pages.

- Role of the migration on evolutionary equilibria

  Analysis in the case of two patches with strong migration between them.

  Case study: S. Mirrahimi. Adaptation and migration of a population between patches. Discr. Cont. Dyn. Syst. B 18 (2013), 753–768.

• Examples of synthesis between ecology and evolution

- Migration strategies and adaptation response to keep pace of a climatic change.

  Case studies: M. Alfaro and J. Coville. Travelling waves in a nonlocal reaction-diffusion equation as a model for a population structured by a space variable and a phenotypic trait. Comm. Partial Diff. Eq. 38 (2013), 2126–2154.

  S. Mirrahimi and G. Raoul. Dynamics of sexual populations structured by a space variable and a phenotypical trait. Theor. Pop. Biol. 84 (2013), 87–103.

- Dispersal evolution in a species invasion front

  Case studies: O. Benichou et al. Front acceleration by dynamic selection in Fisher population waves. Phys Rev. E 86/041908 (2012), 5 pages.

  E. Bouin and V. Calvez. Travelling waves for the cane toads equation with bounded traits. Nonlinearity 27 (2014), 2233–2253.

  V. Calvez et al. Non-local competition slows down front acceleration during dispersal evolution, `https://arxiv.org/abs/1810.07634`, 54 pages.