Partial Differential Equations, Analysis and Use

This program aims to prepare students for research in the field of theoretical and numerical analysis of problems involving partial differential equations (PDEs). It has three components:

- Refresher Courses in the first 2.5 weeks aimed at ensuring a common knowledge base for students from various mathematical backgrounds. These courses are optional but very strongly advised.

- three Basic Courses which offer a broad introduction to the analysis techniques of large classes of partial differential equations and the discrete approximation of their solutions.

- three Advanced Courses on subjects closely related to current research: the analysis of Navier-Stokes equations in Fluid Mechanics, the use of EDPs from biological sources, and the analysis of these models, and finally, an introduction to statistical mechanics and the origin of gases.

**Refresher Courses**

- *Basic tools of functional analysis*, Laure Saint-Raymond, 16h
- *Stochastic tools*, Charles-Edouard Bréhier, 10h
- *Starting with PDEs*, Petru Mironescu, 16h

**Basic Courses (24h)**

- *Variational methods and elliptic PDEs*, Olivier Druet
- *Evolution equations*, Francesco Fanelli
- *Approximation by PDEs*, Julien Vovelle

**Advanced Courses (24h)**

- *Scaling limits and kinetic theory*, Sergio Simonella
- *Mathematical methods for the Navier-Stokes equations*, Lorenzo Brandolese
Refresher courses

Basic tools of functional analysis, Laure Saint-Raymond, 16h

Contents

• Duality : Hahn-Banach theorem, weak and weak* topologies, Lebesgue spaces (3h)
• Distributions : weak derivatives, convolution, regularization (2h)
• Fourier transform (2h)
• Sobolev spaces : embeddings, extension and traces, compactness (3h)
• Weak solution of PDEs : notions of weak solution, a priori estimates, stability (3h)
• Compactness : compensated compactness, averaging lemma, hypoellipticity (3h)

Stochastic tools, Charles-Edouard Bréhier, 10h

• Discrete time martingales; stopping theorems and convergence. Extensions for continuous time martingales.
• Construction of Brownian Motion. Regularity of trajectories.
• Some properties of the Brownian trajectories. Connection with the heat equation.

Starting with PDEs, Petru Mironescu, 16h

Contents

• Geometric aspects of integration theory. Area, co-area, jacobians
• The Bochner integral
• Comparison principles for first and second order partial differential equations. A priori estimates
• Perron’s method. A potential theoretic point of view of smoothness
• The energy method. Uniqueness and domain of influence
• Semi-group methods in the study of evolution equations. Existence, smoothness and additional smoothness
Variational methods and elliptic PDEs
Minimal and constant mean curvature surfaces
Advanced course, Olivier Druet

**Aim of the course.** introduction to classical tools in elliptic PDEs and variational calculus through the study of minimal and constant mean curvature surfaces.

**Contents**

- Local theory of curves and surfaces.
- Schauder elliptic theory: Plateau’s problem for minimal graphs.
- De Giorgi-Nash-Moser elliptic theory: Plateau’s problem for parametrized disks. Tools: regularity theory, Sobolev spaces, ...
Evolution equations
Advanced course, Francesco Fanelli

Contents

• Introduction
classification of PDEs, basic definitions
evolution PDEs: well-posedness, weak and strong solutions

• Transport equations
characteristics
Solutions in Lebesgue and Sobolev spaces
back to characteristics: introduction to Hamilton-Jacobi equations and conservation laws

• Parabolic equations
heat equation: fundamental solution, maximum principles, energy methods
the general case: existence of weak solutions, regularity, uniqueness; maximum principles and Harnack inequality
a non-linear example: the QG equation

• Littlewood-Paley theory
Littlewood-Paley decomposition
Besov spaces
Elements of paradifferential calculus: paraproducts, paralinearization

• Hyperbolic equations and systems
wave equations with regular coefficients
first-order hyperbolic Friedrichs symmetrizable systems
quasi-linear equations and symmetric systems

• Introduction to asymptotic analysis
back to the QG equation: long-time behaviour
introduction to the study of fluids in fast rotation
Approximation by PDEs

Advanced course, Julien Vovelle

In this course, we will see how to understand and describe the large scale limit of various discrete evolution systems (random and deterministic) with the help of partial differential equations. This will be the occasion to use and discover some standard tools from the theory of PDEs, of numerical analysis, and of statistical physics.

Contents

- Martingale in continuous time
- Discrete conservation laws, systems of interacting particles and their asymptotic description by PDEs
- Interacting particles systems
  Independent random walk
  Model of random interface
- Discrete conservation laws (Finite Volume method)
  Parabolic equations
  Hyperbolic equations
  The case of the linear transport equation: optimal convergence estimate by two different methods (deterministic/probabilistic).
The kinetic theory of rarefied gases, initiated by Boltzmann, realizes the program of the rigorous connection between macroscopic effective equations and interacting models at the atomic scale.

The object of this course is to introduce a basic formalism in statistical mechanics, present the fundamental results of the theory for systems far from equilibrium, and finally the open problems of current interest.

Contents

• Equilibrium systems
  Thermodynamic system
  Statistical ensembles
  Dilute gas, correlation functions
  Thermodynamic limit (notions)

• Kinetic theory
  Scaling limits: Vlasov, Boltzmann, Landau
  Lorentz gas, hard spheres: law of large numbers and Boltzmann equation
  Hierarchical method
  Implications of the Hewitt-Savage theorem
  Irreversibility, entropy
  Central limit theorem, fluctuating Boltzmann equation.
Mathematical methods for the Navier-Stokes equations
Advanced course, Lorenzo Brandolese

This course is devoted to one of the fundamental models of fluid mechanics, the Navier-Stokes equations. We will present several methods for studying such equations (compactness arguments, fixed point methods, dyadic decompositions, Fourier splitting device, bilinear estimates, etc.).

Besides the classical questions of the existence of weak solutions and of unique strong solutions, we will study in some details the problems related to the large time and the far field behavior of the flows.

The last part of the course will be a quick introduction to more complicated models in fluid mechanics, where the equation of the velocity field is coupled with other physical quantities (temperature, magnetic field, etc.).

Contents

• Finite energy solutions
  Leray’s weak solutions.
  About the regularity of weak solutions.
  Energy dissipation and Fourier splitting method.
  Asymptotic profiles as \( t \to +\infty \).

• Mild solutions
  Kato’s solutions and their uniqueness in critical spaces.
  The limit case: the Koch-Tataru theorem and norm inflation phenomena.
  Finite time blowup for a Navier-Stokes like equation.
  Self-similar solutions.

• Asymptotic behavior as \( |x| \to +\infty \).
  Miyakawa’s theorem.
  Point-wise upper and lower bounds.

• Introduction to some other model in fluid mechanics
  Geophysical flows and the Boussinesq approximation. Magnetohydrodynamics.
The aim of this course is to investigate several Partial Differential Equations models with applications to spatial ecology and evolutionary biology. The common topic will be the description of a population with respect to various structural variables: space, age, phenotypic trait.

The course will address a balance of old and new problems. Focused applications are: species invasion, collective motion, quantitative genetics. Mathematical tools will be borrowed from previous analysis and PDE courses. Specific tools will be also developed during the course.

• Reaction-diffusion equations in spatial ecology

  – Fisher’s equation and propagation phenomena
    Existence of traveling waves; spreading speed for the Cauchy problem
  – Kinetic modeling of species propagation
    Existence of traveling waves; spreading speed for the Cauchy problem; wave acceleration

• Reaction-diffusion equations in evolutionary biology

  – Mutation-selection equilibria and concentration phenomena
    Asymptotic analysis in the regime of small variance; convergence towards the viscosity solution of a Hamilton-Jacobi equation; existence and uniqueness
- Fisher’s infinitesimal model
  Asymptotic analysis in the regime of small variance; hydrodynamic limit

- Role of the migration on evolutionary equilibria
  Analysis in the case of two patches with strong migration between them.

Examples of synthesis between ecology and evolution

- Migration strategies and adaptation response to keep pace of a climatic change.

- Dispersal evolution in a species invasion front