Geometry, h-principle and effective methods

This program aims to give a wide panorama of problems and methods in Differential Geometry, and of the related PDEs. It consists in two Basic Courses and two Advanced Courses.

Basic Courses (24h)

- h-principle, Vincent Borrelli and Jean-Claude Sikorav
- Riemannian geometry (towards isometric immersions), Abdelghani Zeghib

Advanced Courses (24h)

- Effective Methods in Geometry, Francis Lazarus and Boris Thibert
- Besse and Zoll manifolds, Marco Mazzucchelli.

The course on h-principle constitutes an introduction to the ideas of M. Gromov about solving PDEs using homotopical methods. These equations occur mostly in Geometry, especially isometric embeddings, but in the last the have been spectacular applications of these ideas to Fluid Mechanics.

The course of Riemannian Geometry will mostly focus on questions related to submanifolds of Riemannian manifolds, especially the Euclidean space, and on the problem of isometric immersion (or embedding).

The course Effective Methods in Geometry is rather unusual in aiming to show how abstract mathematics can be translated into concrete computations. A wide set of methods will be presented, ranging from combinatorial ones to optimal transport. Many examples will be given: isometric embeddings, algorithmic knot theory, word problems and computational topology, optical lenses...

The course Bess and Zoll Manifolds will present fascinating recent developments on a classical problem of Riemannian geometry: for which closed manifolds are all geodesics circles, and more particularly all of the same length? It will also introduce to the burgeoning new field of Systolic contact geometry.

h-principle

Basic course, Vincent Borelli and Jean-Claude Sikorav

The *h*-principle, where the ”h” stands for ”homotopy”, was created by M. Gromov in the 1970s, its origins being the works of J. Nash (and N. Kuiper) on isometric embeddings in the 1950s, and those of S. Smale (1958) on the immersions of spheres and of A. Philipps (1960s) on foliations. The main idea is that many partial differential equations (and more generally partial differential relations, ie. inequalities) can be solved using homotopical methods. These equations occur mostly in Geometry (eg isometric embeddings, curvature prescription), but in the last the have been spectacular applications of these ideas to Fluid Mechanics.

The main part of this course will be devoted to the method of Convex Integration, invented by Gromov in order to determine the presence of a h-principle and to allow for the resolution of large families of differential relations. This method has been fundamentally renewed in the last twenty years, with ideas making it much more concrete. In this course, we will see how it applies to the classification of immersions or to the construction of explicit C^1 isometric embeddings.

A smaller part of the course will be devoted to other methods (removal of singularities, invariant differential relations). We shall mostly focus on applications to symplectic and contact topology, which have seen a lot of progress lately.

Contents

- The theorem of Whitney-Graustein
- Corrugations in the way of Thurston
- The theorem of Smale-Hirsch
- Gromov’s theorem on ample relations
- Nash-Kuiper’s theorem
- Explicit constructions of C^1 isometric embeddings
- Removal of singularities
- Invariant differential relations.

References

- D. Spring, *Convex Integration Theory*, Monographs in Mathematics, vol 92
- Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Grad. Studies in Math., Amer. Math. Soc., 2002.
- M. Gromov, *Partial differential relations*, Ergebnisse der Math., Springer, 1986.
- V. Borrelli, S. Jabrane, F. Lazarus, and B. Thibert, *Isometric embeddings of the square flat torus in ambient space*, Ensaios Matemáticos 24. Soc. Brasil. de Matemática, 2013.

Riemannian Geometry (towards isometric immersions)

Basic course, Abdelghani Zeghib

After reviewing the main objects of Riemannian Geometry, this courses focuses on questions related to submanifolds or Riemannian manifolds, especially (but not only) of the Euclidean space, and on the problem of isometric immersion (or embedding).

Contents

- Riemannian metrics, curvatures
- Extrinsic geometry of submanifolds
- Surfaces in 3-Euclidean space
- Rigidity and existence in the elliptic case, Prescribing curvature problems
- Non-existence in the hyperbolic case
- Developable surfaces
- Generalization to higher dimensional Euclidean space
- Generalization to constant curvature spaces
- The isometric immersion problem, Nash Theorems
- Proof in the compact smooth case
- Pseudo-Riemannian case.

References

- M.P. do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.
- Q. Han and J.-X. Hong, *Isometric embedding of Riemannian manifolds in Euclidean spaces*, Amer. Math. Soc. 2006.
- M. Spivak, *A comprehensive introduction to differential geometry*, 5 volumes, Publish or Perish, 1979.

Effective methods in Geometry

Advanced Course, Francis Lazarus and Boris Thibert

This course aims at showing how abstract mathematics can be translated into concrete computations. A first step in this translation requires to discretize the studied problem that may come from mathematics itself or from other fields such as physics. We shall see on various examples how to perform this discretization and how to solve the corresponding formulation.

A first part will focus on combinatorial aspects issued from topology and geometry, where the underlying mathematical problems have a discrete nature. We will start with the isometric embedding of the flat torus in order to explore PL isometric embeddings as well as other questions mixing topology and geometry.

A second part will focus on numerical aspects. We shall concentrate on some equations of Monge-Ampère type that intervene in various fields such as geometry, optimal transport or optics. We will show how the geometrical discretization of the theory of optimal transport leads to solving optimization problems. In turn, this approach may lead to the realization of concrete objects, such as optical lenses.

Contents

- The theorem of Burago and Zalgaller for PL isometric embeddings of surfaces endowed with polyhedral metrics.
- Effective aspects of the problem of the existence of topological embedding.
- Algorithmics of knot theory.
- Effective computations in the fundamental group of a surface.
- What we cannot compute: how the undecidability of the word problem impacts computational topology.
- Theory of optimal transport: Monge problem, Kantorovitch relaxation, Kantorovitch duality, notion of c-conjugate, the Wasserstein distance, Mac Cann interpolation.
- Optimal transport for discrete measures: assignement problems, auction algorithm, entropic regularization.
- Optimal transport between a measure with density and a discrete measure: connection with computational geometry (Voronoi diagrams, Laguerre diagrams, ...), the Oliker-Prussner algorithm, concave formulation and Newton's methods.
- Applications to inverse problems in anidolic [or nonimaging] optics.

References

- J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer, 1995.

- J. Hass, J.C. Lagarias, N. Pippenger, *The Computational Complexity of Knot and Link Problems*, J. ACM 46 (1999), 185-211.
- V. Despré and F. Lazarus, *Computing the Geometric Intersection Number of Curves*, J. ACM, to appear.
- J. Kitagawa, Q. Mérigot, and B. Thibert, *Convergence of a Newton algorithm for semi-discrete optimal transport*, accepted to J. Europ. Math. Soc., `https://arxiv.org/abs/1603.05579` (2016).
- F. Santambrogio, *Optimal transport for applied mathematicians*, Birkhäuser, 2015.
- J. Meyron, Q. Mérigot, B.Thibert, *Light in Power: A General and Parameter-free Algorithm for Caustic Design*, ACM Transactions on Graphics 2018.

Besse and Zoll Manifolds

Advanced Course, Marco Mazzucchelli

On a closed manifold, a Riemannian metric is called *Besse* when all of its geodesics are closed (i.e. embedded circles). If the geodesics are also assumed to all have the same length, the metric is called *Zoll*. These classes of manifolds have been thoroughly investigated since the beginning of the 20th century. The following are some of the questions concerning them:

1) Which manifolds admit a Besse or Zoll Riemannian metric?

2) Can a given manifold admit several non-isometric Zoll Riemannian metrics?

3) Are Besse metrics on certain closed manifolds necessarily Zoll metrics?

4) Can one characterize the Zoll metrics in terms of the prime length spectrum (i.e. the set of lengths of the non-iterated closed geodesics)?

In this generality, these questions are still open. Some answers to questions 1-2 have been provided in celebrated results due to Bott-Samelson and Guillemin. Question 3 was answered positively for the 2-sphere by Gromoll-Grove, and very recently for all spheres of dimension at least 4 by Radeschi-Wilking. Question 4 has been answered positively for the 2-sphere by Mazzucchelli-Suhr, who also proved results going in this direction for certain higher dimensional manifolds. In this course, after introducing the needed background on the geodesic flow, we will go over these results. At the end of the course, we will go beyond the geodesic dynamics, and briefly introduce the Zoll contact forms and their relations with contact systolic geometry.

Contents

- Review of the geodesic flow
- Morse-Bott theory for geodesics
- Besse and Zoll Riemannian manifolds
- On question 1: Bott-Samelson Theorem
- On question 2: Guillemin Theorem
- On question 3: Radeschi-Wilking Theorem
- On question 4: min-max characterizations of Zoll Riemannian metrics
- Review of contact geometry
- The Boothby-Wang construction
- Contact systolic ratio and Zoll contact forms

References

- A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Math. 93, Springer, 1978.

- H. Geiges, *An introduction to contact topology*, Cambridge Studies in Adv. Math. 109, 2008.
- J. Milnor, *Morse theory, Based on lecture notes by M. Spivak and R. Wells.* Annals of Math. Studies 51, Princeton Univ. Press, 1963.
- M. Mazzucchelli and S. Suhr, *A characterization of Zoll Riemannian metrics on the 2-sphere*, `https://arxiv.org/abs/1711.11285` , to appear in Bull. Lond. Math. Soc., 2017.
- M. Mazzucchelli and S. Suhr, *A min-max characterization of Zoll Riemannian metrics*, `https://arxiv.org/abs/1809.08689`
- M. Radeschi and B. Wilking, *On the Berger conjecture for manifolds all of whose geodesics are closed*, Invent. Math. 210 (2017), 911–962.
- A. Abbondandolo, B. Bramham, U. L. Hryniewicz and P. A. S. Salomão, *Sharp systolic inequalities for Reeb flows on the three-sphere*, Invent. Math. 211 (2018), 687-778.